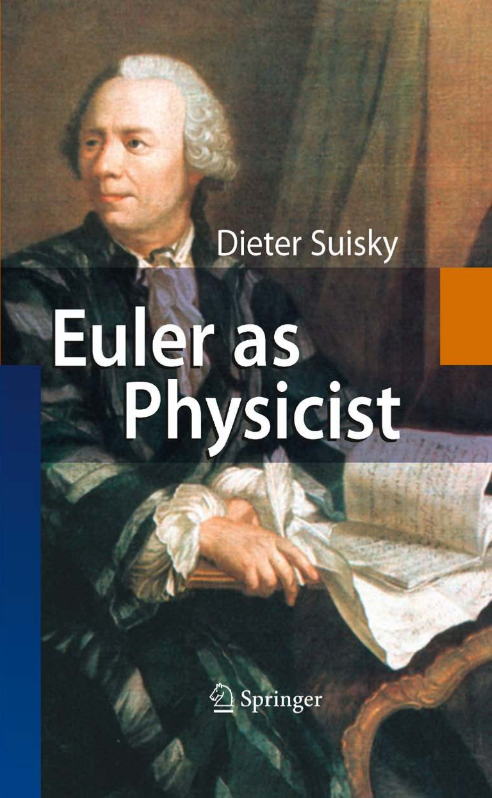

Dieter Suisky

Euler as Physicist

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Preface

In this book the exceptional role of Leonhard Euler in the history of science will be analyzed and emphasized, especially demonstrated for his fundamental contributions to physics. Although Euler is famous as the leading mathematician of the 18th century his contributions to physics are as important and rich of new methods and solutions. There are many books devoted to Euler as mathematician, but not as physicist.

In the past decade, special attention had been directed at the development of science in the 18th century. In three distinguished tercentenary celebrations in occasion of the births of Pierre Louis Moreau de Maupertuis (1698–1759), Emilie du Châtelet (1706–1749) and Leonhard Euler (1707–1783) the merits of these scholars for the development of the post-Newtonian science had been highly acknowledged. These events were not only most welcome to remember an essential period in the past, but are also an opportunity to ask for the long lasting influence on the further development of science until present days.

Euler's contributions to mechanics are rooted in his program published in two volumes entitled *Mechanics or the science of motion analytically demonstrated* very early in 1736. The importance of Euler's theory results from the simultaneous development and application of mathematical and physical methods. It is of particular interest to study how Euler made immediate use of his mathematics for mechanics and coordinated his progress in mathematics with his progress in physics. Euler's mechanics is not only a model for a consistently formulated theory, but allows for generalizations of Euler's principles.

Though his pioneering work on mechanics had an essential influence in the 18th century, its impact on the 19th century was obscured by the overwhelming success of his mathematical writings. Euler anticipated Mach's later criticism of absolute motion and Einstein's assumption on the invariance of the equation of motion in inertial systems. It will be demonstrated that even problems in contemporary physics may be advantageously reconsidered and reformulated in terms of Euler's early unified approach. The interplay between physics and mathematics which appeared in the 18th century will be compared to the development of physics in the 20th century, especially to the development of quantum mechanics between 1900 and 1930.

The reader of Euler's works benefits from his unique ability to preserve mathematical rigour in the analytical formulation of physical laws. The principles and methods found in the original sources may be advantageously compared to later developments, interpretations and reformulations. The extraordinary power of Euler's program and legacy is due to the successful frontier crossing between different disciplines presented in an exemplarily clear terminology.

I am grateful to those individuals who draw my attention to Euler's original work and its contemporary interpretation.

I would like especially to thank Dr. Hartmut Hecht (Berlin-Brandenburgische Akademie der Wissenschaften), Dr. Peter Enders (Berlin), Dr. Peter Tuschik (Humboldt University, Berlin), Dr. Rüdiger Thiele (University of Leipzig), Prof. Robert E. Bradley (Adelphi University, Garden City), Prof. C. Edward Sandifer (Western Connecticut State University), Prof. Heinz Lübbig (Berlin), Prof. Eberhard Knobloch (Technical University Berlin), Prof. Ruth Hagengruber (University of Paderborn), Dr. Andrea Reichenberger (University of Paderborn) and Prof. Brigitte Falkenburg (University of Dortmund) for many stimulating and encouraging discussions.

In particular, I am indebted to Prof. Wolfgang Nolting (Humboldt University, Berlin) for stimulating support over a period of years.

Finally, I would like to thank my wife Ingrid Suisky for rendering every help in preparing the manuscript.

Berlin *Dieter Suisky*

Introduction

Read Euler, read Euler, he is the master of us all.
Laplace

Euler began his extraordinary scientific life in the third decade of the 18th century and spent most of his career working in St. Petersburg and Berlin.[1] His first paper was published in 1726, the last paper of his writings only in the mid of the 19th century. For his scientific career he was well prepared and educated by Johann Bernoulli who introduced him in mathematics by personal supervision[2] since 1720. His close relations to the Bernoulli family, especially to the older sons of Johann Bernoulli, let him finally settle in St. Petersburg[3] in 1727.

In this time, the scientific community was involved in controversial debates about basic concepts of mechanics like the nature of space and time and the measure of living forces, the priority in the invention of the calculus and the application of the calculus to mechanics. The promising Newton-Clarke-Leibniz debate between 1715 and 1716 was truncated by the death of Leibniz in 1716. The letters on the basic principles of natural philosophy exchanged by him with Clarke published in 1717 [Leibniz Clarke] can be considered as a collection and listing of solved and

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[1] Euler lived from 1707 to 1727 in Switzerland. In Basel, he studied mathematics under the supervision of Johann Bernoulli (1667–1748). From 1727 to 1740 Euler worked in St. Petersburg. Euler grew up together with Daniel Bernoulli (1700–1782), Nikolaus II Bernoulli (1695–1726) and Johann II Bernoulli (1710–1790) Bernoulli, the sons of Johann Bernoulli. The change of Euler to St. Petersburg was mainly stimulated by the leave of Daniel Bernoulli who went to St. Petersburg as a professor of mathematics in 1725. After a successful and fruitful period in St. Petersburg Euler spent the decades from 1741 to 1766 in Berlin before he went back to St. Petersburg (1766–1783). Essential contributions of Euler to mechanics were published during Berlin period between 1745 and 1765.

[2] "A. 1720 wurde ich ich bey der Universität zu den Lectionibus publicis promovirt: wo ich bald Gelegenheit fand, dem berühmten Professori Johanni Bernoulli bekannt zu werden, welcher sich ein besonderes Vergnügen daraus machte, mir in den mathematischen Wissenschafften weiter fortzuhelfen. Privat Lectionen schlug er mir zwar (...) ab: er gab mir aber (...) alle Sonnabend Nachmittageinen freyen Zutritt bey sich, und hatte die Güte mir die gesammlete Schwierigkeiten zu erläutern, (...)." [Fellmann (Row)] Euler dictated the autobiography to his son Johann Albrecht in the second Petersburg period.

[3] "Um dieselbige Zeit wurde die neue Akademie (...) in St. Petersburg errichtet, wohin die beyden ältesten Söhne des H. Johannis Bernoulli beruffen wurden; da ich denn eine unbeschreibliche Begierde bekam mit denselben zugleich A. 1725 nach St. Petersburg zu reisen." [Fellmann (Row)]

unresolved problems in mechanics and philosophy. One of the crucial points is the debate on the nature of time, space and motion. This listing will be completed by the problems originating from the controversy on the priority of the invention of the calculus which had been initiated by British scientists in 1690's and continued until 1711 [Meli]. Furthermore, the scientific community was involved in the long-lasting debate on the true measure of living forces and, as a consequence, split into different schools. The post-Newtonian scientists were confronted with a legacy rich of new methods and solutions of old problems, but also rich of open questions and had to struggle with a collection of most complicated problems which were, moreover, closely interrelated preventing from the very beginning a merely partial solution. Nevertheless, such partial solutions were constructed by the several schools which had been appeared, established and settled preferentially in France, England and Germany.[4] The debate was dominated by these schools, the Cartesians, the Newtonians and the Leibnizians, respectively. The common problem was to find a way out from the merging of scientific and non-scientific problems.

All the scientists were involved in the controversies as Euler reported 30 year later [Euler, E343].[5] The leading scientist decided it is necessary to reconsider the legacy of their predecessors without prejudice, i.e. independently of the country where it was produced (compare [Châtelet, Institutions]). Voltaire did a great work to accustom people in France to the ideas of Newton [Voltaire, Élémens]. Wolff was interesting in that his system becomes popular in France.

The profound education Euler received from Johann Bernoulli is not the only long lasting influence on his thinking and life. There is also the personal and scientific relation to Daniel Bernoulli which was of great influence on the scientific biography of Euler. The concept of his program for mechanics published in the *Mechanica* may be related to the discussion on the status of the basic law of mechanics which had been later called *Principe général et fondamental de toute la mécanique* by Euler [Euler E177]. Euler referred in the *Mechanica* [Euler E015/016, § 152] to the relations between the change of velocity and forces proposed by Daniel Bernoulli published in 1722 [Bernoulli Daniel].[6] Euler derived a general

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[4] In Germany the scientific discussions were dominated by the Leibniz-Wolffian school. Wolff (1679–1754) reinterpreted the Leibnizian system of monadology in terms of least elements of bodies. Euler criticized the *Lehrgebäude von den Monaden* [Euler E081] distinguishing carefully between the original Leibnizian theory and the Wolffian interpretation. However, judging on the merits of the predecessors we have to take into account that most of the writings of Leibniz and Newton were unpublished in 18th century. Therefore, the followers had to have to reconstruct and re-invent the original versions from the parts which were published. As it can be demonstrated for Leibniz's contributions to logic which had been only rediscovered by Couturat and Russell in 19th century, this is a very complicated procedure.

[5] Euler commented on the debate on the nature of light [Euler E343, Lettre XVVII–XXI], the gravitation [Euler E343, Lettre XLV, LVI–LXVIII], the system of monads and the origin of forces [Euler E343, Lettre LXIX–LXXIX], Leibniz's metaphysics and other philosophical systems [Euler E344 Lettre LXXX–XCIX, CXXII–CXXXII].

[6] "152. Apparet igitur non solum verum esse hoc theorema, sed etiam necessario verum, ita ut contradictionem involveret ponere $dc = p^2 dt$ vel $p^3 dt$ aliamve functionem loco p. Quae omnes cum Clar. Dan. Bernoullio in Comment. Tom. I. aeque probabiles videantur, de rigidis harum propositionum demonstrationibus maxime eram sollicitus." [Euler E015/016, § 152]

relation between the change of motion, i.e. the change of velocity, and the forces impressed upon the body. Moreover, Euler claimed that this relation is not only "valid" (non solum verum), but also "necessarily valid" (sed etiam necessario verum). In 1743, treating the same relation between the acceleration and forces as a definition, d'Alembert referred to Daniel Bernoulli and Euler and claimed that he is not willingly to decide the question whether this principle is only justified by experience or a necessary truth [d'Alembert, Traité, § 19].

The necessity of a mechanical law is confirmed by a comparison to mathematics, especially to geometry [Euler E181, §§ 1 and 2]. Euler preserved this methodological principle and repeated this approach to mechanics from 1734 also almost 30 years later in the *Lettres à une princesse d'Allemagne* [Euler E344, Lettre LXXI][7] and in the *Theoria motus corporum solidorum seu rigidorum* [Euler E289]. The distinction between necessary statements and contingent statements had been elaborated by Leibniz who based his program for mechanics on the assumption that geometry has to be completed by principles which explain the action and the suffering of bodies [Leibniz, Specimen, I (11)]. The geometrical truths are considered as necessary truths, in contrast to the laws in mechanics or other sciences related to experience which had been characterized as contingent truths [Leibniz, Monadology, §§ 31–36]. As a direct consequence it follows that, although mechanics is related to experience the theory has to be based on principles of the same reliability (or necessity) as mathematics.[8] The program for mechanics has to be supported by the *transfer* of mathematical principles to mechanics. Then, the mathematical principles are not softened or violated,[9] but form a constitutive part of the theory. As a consequence, mathematical and mechanical principles had been not only applied or transferred into the other discipline, but preferentially confirmed and mutually tested in their reliability and applicability. Newton demonstrated this program for the transfer of geometrical principles to mechanics.

After the invention of the calculus by Newton and Leibniz, the same problem arises for the transfer of the arithmetical principles of the calculus to mechanics. However, even for the inventors of the calculus, this step can be no means taken for granted since it was almost automatically necessary to accept the earlier Cartesian program and basic principles of Cartesian methodology. Newton stated:

> Men of recent times, eager to add to the discoveries of the Ancients, have united the arithmetic of variable with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output, but the advance is less of a blessing if you look at the complexity of the conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout

[7] "Quelque fondée qui soit cette loi, qui pourroit aller de pair avec les vérités géométriques, (...)." [Euler E343, Lettre LXXI]

[8] The mathematical methods are to be developed in conformity with the rigor known from the legacy of the Ancients (compare Chaps. 2, 3, 4 and 5).

[9] This program has been invented and demonstrated by Newton. It can read off from the title of his treatise *Philosophiae naturalis principia mathematica* [Newton, Principia]. It is very elucidating to take notice from Euler's program presented in the titles *Mechanica sive motus scientia analytice exposita* [Euler E015/016] and *Theoria motus corporum solidorum seu rigidorum* [Euler E289].

> way quantities which in geometry were designated by the drawing of a single line. [Mathematical Papers of Isaac Newton 4:421]

Euler renewed the Cartesian program. In 1727, Euler composed an analytical foundation of the calculus [Euler 1727] whose basic principles had been preserved in the following decades [Euler E212]. Stimulated and educated by his supervisor Johann Bernoulli, Euler could made immediate use of the transmitted results of the examination and application of the new method by Leibniz and Johann Bernoulli since 1684.[10] Analyzing Euler's philosophical statements people claimed that Euler renewed the Cartesian dualism between body and soul. However, comparing Euler's basic assumptions on the nature of bodies with the original Cartesian version, a remarkable and strong difference can be observed. Euler introduced an important change of Descartes concept of bodies. Descartes claimed that the extension is basic properties of all bodies, stating *res extensa sive corpus*. Euler introduced a program for mechanics based on the concept of bodies of infinitesimal magnitude.

> Those laws of motion which a body observes when left to itself in continuing rest or motion pertain properly to infinitely small bodies, which can be considered as points. (...) The diversity of bodies therefore will supply the primary division of our work. First indeed we shall consider infinitely small bodies (...). Then we shall attack bodies of finite magnitude which are rigid. [Euler E015/016, § 98]

Reading Euler's program and considering the actual background formed by the state of affairs in mechanics and mathematics in the first half of the 18th century, it follows that Euler's program for mechanics should be necessarily based on an additional program, a program for reinterpretation, application and development of the calculus invented by Newton and Leibniz. In 1734, this program was widely hidden and only explicitly formulated in the treatise *Institutiones calculi differentialis* [Euler E212] written in 1748 and presented only in 1755. This relation can be confirmed by the reference to a statement in the Preface of *Mechanica*. Here, Euler stated that the shortcoming of the geometrical method is the lack of an *algorithm* which can be used for the modelling and calculation of problems which deviated only slightly in some detail from the standard formulation and solution.[11]

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[10] Similarly, in the 20th century, Heisenberg could made use of the discussions between Planck, Bohr, Einstein and Sommerfeld on the foundation of quantum mechanics (compare Chap. 8). The supervisors had established the new discipline and prepared the problem for solution, but the decisive step beyond the commonly accepted classical frame was done by the young Heisenberg [Heisenberg 1925]. Similarly to Euler who decided to compose mechanics without geometry, but solely analytically demonstrated, Heisenberg decided to chose a "mechanics without positions and paths" [Heisenberg 1925] making only use of the methods of "transcendental algebra" [Schrödinger, Second Announcement]. Heisenberg rejected the "positions and paths", Pauli discarded the "causality".

[11] Moreover, Euler invented algorithm also for the every day applications of mathematics. "Man pflegt nämlich mit der eigentlichen Arithmetik noch einige Regeln, welche in der allgemeinen Analysi oder Algebra ihren Grund haben, zu vereinigen, damit ein Mensch, welcher dieselbe erlernet, auch im Stande sei, die meisten Aufgaben, so in dem gemeinen Leben vorzufallen pflegen, aufzulösen, ohne in der Algebra geübt zu sein." [Euler E017 Chap. 1, § 1] This book was written in parallel with the *Mechanica* in 1735 and published in 1738. Obviously, the main principle is to invented *algebraic methods* both for the solutions of problems in science and every day life.

> That which is valid for all the writings which are composed without the application of analysis is especially true for the treatises on mechanics $(^*)$. The reader may be convinced of the truths of the presented theorems, but he did not attain a sufficient clarity and knowledge of them. This becomes obvious if the suppositions made by the authors are only slightly modified. Then, the reader will hardly be able to solve the problems by his own efforts if he did not take recourse to the analysis developing the same theorem using the analytical method. [Euler E015/016, Preface] $(^*)$ Newton's *Principia* (1687)

The complete title of Euler's book on mechanics is simultaneous an abbreviated, but precise program for the application of the calculus in mechanics: *Mechanica sive motus scientia analytice exposita*, i.e. mechanics or the science of motion demonstrated by means of analytical methods or the application of the calculus.[12] Here, the analytical approach is opposed to the geometrical treatment[13] preferred by Newton in the *Principia*. Following Descartes and Newton, Euler constructed mechanics on the basic concept of rest and motion and, methodologically on the basic distinction between *internal* and *external* principles.

Euler's program for mechanics is simultaneously also a program for mathematics underlying the analytical approach. This program had been developed in the treatise entitled *Calculus differentialis* written in 1727, but only published form the first time in 1983 [Euler 1727], [Euler, Juschkevich]. By this approach, Euler continued a tradition which can be traced back to his predecessors Descartes, Newton and Leibniz[14] to develop simultaneously mathematics and mechanics with a preference to the mathematically established algorithms. However, there is an essential difference. Although Newton invented an arithmetical algorithm [Newton, Method of Fluxions], [Newton, Principia], he preferred geometrical methods for the confirmation of the analytically obtained results and specified, e.g. the independent variable to be related to a "continuous flux" or "time". An alternative foundation based on arithmetic had been discussed by Leibniz [Leibniz, Historia] who based the method on the correlated operations formed by differences and sums [Leibniz, Elementa],[15] where the first operation results in a differentiation whereas the second one results in the inverse operation, called integration.[16] From the very

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[12] The program is related to the foundation of the calculus in terms of algebraic operations which had been later developed by Euler [Euler E212]. The father of such type of foundation is Leibniz who invented the operations of the calculus as operation rules applied to quantities [Leibniz, Nova Methodus].

[13] The Eulerian program had been later continued and developed by Lagrange. The title of his basic book on mechanics is *Mécanique analytique* [Lagrange, Mécanique]. Lagrange stressed that he did not make use of any figures.

[14] It seems to be reasonable to include further Archimedes, Galileo and Huygens. Euler referred for the explanation of the conservation of state to Archimedes' consideration on the equilibrium between bodies [Euler, E015/016 § 56].

[15] Leibniz considered both the operation from the very beginning, he invented also the formal rules and the signs for the different operation and called the procedure calculus of differences and summation, although the name integration was finally due Johann Bernoulli.

[16] Compare the appendix to the Varignon's letter to Leibniz from May 23, 1702, entitled *Justification du Calcul des infinitesimales par celuy de l'Algebre ordinaire* [Leibniz, Mathematische Schriften, vol. 4, pp. 99–106].

beginning, Euler preferred arithmetical methods in the foundation of the calculus [Euler 1727], [Euler E212].

The geometry independent *analytical* approach to mathematics had been later completed by Cauchy, Bolzano and Weierstraß. After the rigorous foundation of the calculus, the limit interpretation of differentials had been re-implemented into mechanics [Helmholtz, Vorlesungen], [Klein, Elementarmathematik]. Although mechanics had been formerly based on a questionable mathematical background by those methods which were condemned by the partisans of the 19th century foundation, all basic relation derived by Euler [Euler E015/016] were preserved. None of the basic laws of mechanics was susceptible to a reformulation in terms of limits.[17] In contrast to the formulation of mechanics in terms of differentials by Euler resulting in an essential progress in comparison to all earlier representations by his predecessors, the reformulation of Eulerian analytical mechanics in terms of differential quotients does not result in new insights in the basic principles of mechanics, but was preferentially of importance for mathematics.

> Parallel mit der (...) Entwicklungsreihe, auf der sich die heutige wissenschaftliche Mathematik aufbaut, hat sich eine wesentlich verschiedene Auffassung der Infinitesimalrechnung durch die Jahrhunderte fortgepflanzt. Sie geht zurück: 1. auf *alte metaphysische Spekulationen über den Aufbau des Kontinuums* aus nicht mehr weiter zerlegbaren, 'unendlich kleinen' Bestandteilen. Als (...) Beleg nenne ich den Titel des (...) Buches *Cavalieris* 'Geometria indivisibilibus continuorum promota', der seine wahre Grundauffassung andeutet. [Klein, Elementarmathematik, p. 231]

This idea corresponds to measuring procedure established by Galileo. The general relations between time, space and motions are represented by the relations between finite temporal and spatial intervals, but never in terms of the relation between instants. This representation persisted in Minkowski's construction of the "absolute world" represented as the "space-time-union" where separated time or separated space are "mere shadows" [Minkowski, Space and Time], [Minkowski, Raum und Zeit] (see also Pyenson [Pyenson]).

The importance of Euler's theory results from the simultaneous development and application of mathematical and physical methods. Euler continued a practice which had been established by his predecessors Galileo, Descartes, Newton and Leibniz. In this book it will be demonstrated that the development of physics in 19th and 20th centuries can be considered as a natural continuation and completion of the Eulerian program.

The foundation was initially based on the correlation between mathematical language and physical theory, then, the mathematical foundation had been separated from the mechanical models.[18] Finally, Weierstraß [Weierstraß] invented

[17] This problem appeared only after the invention of quantum mechanics and its solution had been elaborated by Heisenberg [Heisenberg 1925], [Heisenberg, Anschaulich]. The other alternative to point-like bodies appeared in the second half of the 20th century. Instead of point-like models for elementary particles, physicists introduced new objects of one-dimensional spatial extension being called "strings". [http://ww.superstringtheory.com/basics/index.html]

[18] "Einem Unterschied dieser neuesten Entwicklung der älteren gegenüber will ich hier sogleich hervorheben: Die (...) Begriffsbildungen sind überwiegend in Hinblick auf die Anwendungen in

a concept appropriate for mathematicians which was later re-implemented and re-imported into physical theory.[19] This foundation runs into trouble as physical quantities had been detected which cannot be represented simultaneously by the Weierstraßian procedure like the position and the momentum of an electron. Then, these quantities are related to each other by Heisenberg's uncertainty relation [Heisenberg, Anschaulich]. Reading Euler's program for mechanics and considering the actual background formed by the state of affairs in mechanics and mathematics in the first half of the 18th century, it follows that Euler's program for mechanics should be necessarily based on an additional program, a program for the reinterpretation, application and development of the calculus invented by Newton and Leibniz.

From the very beginning, the development and application of the calculus was closely connected with the mechanical interpretation of the infinitesimal quantities. This follows from Newton's decision to call these infinitesimal quantities "fluxions" and relate them to other finite quantities, the "fluents", by the invention of an "infinitely small quantity *o*" being related to time, but which is, in contrast to the time measured in experiment, not susceptible to detection.[20] In both case, the correlation to *time* is obvious since the fluents do not preserve their magnitude, but are varying in dependence on time. The relation to the motion of a body which changes its place by motion (compare [Newton, Principia]) is intended and will be confirmed by the distinction between motion and the very first moment of motion when a force is impressed upon the body [Newton, Principia], [Newton, Quadrature].

Leibniz preferred a foundation which is independent of time. The advantage of this approach is readily demonstrated since the infinitesimal quantities are related to each other by rules which follow solely from their specific mathematical properties which are independent of an interpretation within a mechanical model.

A new basis for the modelling and solution of the problem arose from the application of the calculus to mechanics, i.e. the motion of bodies where the finite and infinitesimal quantities are interpreted not only geometrically, i.e. related to distances, but also temporally, i.e. to bodies travelling a certain distance in a certain time.

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der Natur entstanden und ausgebaut worden; man denke nur an den Titel des Fourierschen Werkes! Die neueren (...) Untersuchungen sind aber Produkte rein mathematischen Forschungstriebes, der nicht auf die Bedürfnisse der Naturerklärung bedacht ist, sie haben bisher auch wohl noch keine direkte Anwendung gefunden. Ein Optimist wird natürlich meinen, daβss zweifellos einmal die Zeit für solche Anwendungen kommen wird." [Klein, Elementarmathematik, p. 220]

[19] Compare, e.g. Helmholtz, *Vorlesungen über die Dynamik discreter Massenpunkte* [Helmholtz, Vorlesungen].

[20] Nowadays, Newton's and Leibniz's theories of space and time attracted attention in basic research on the properties of the world (in search of a "correct quantum gravity") as a whole [Smolin] as well as in their relation to quantum mechanics [Wilczek 2004a], [Wilczek 2004b]. Smolin formulated the problem in terms of the difference between "background dependent" and "background independent" approaches: "This leads to a careful statement of what physicist mean when we speak on background independence. (...) *Must a quantum theory of gravity must be background independent, or can there can be a sensible and successful background dependent approach*? (...) *The reason that we do not have a fundamental formulation of string theory, from which it might be possible to resolve the challenge posed by the landscape, is that it has been so far developed as a background dependent theory*." [Smolin]

Although the problem becomes more complicated due to invention of quantities of different type like distances and time intervals the advantage of the procedures becomes evident since some of the previously disturbing ambiguities could now be removed, i.e. in the first step as far as the application to mechanics was concerned. The definition of velocity was either related to the *ratio* of finite distances and finite temporal intervals or to the *ratio* of infinitesimal distances and infinitesimal temporal intervals.

In the original version, Newton related the *phenomena of motion* to the *forces of nature*. Obviously, Leibniz did not essentially modify this program since he also related the *phenomena* to *geometry* and completed geometry by additional principles which are related to the forces[21] [Leibniz, Specimen]. The step beyond the Newtonian program is performed by the assumption of *conservation law* for living forces which is based on the rejection of perpetual motion.[22] The assumption of living forces had been later acknowledged by Euler, d'Alembert, the Bernoullis and Châtelet [Châtelet, Institutions], [Châtelet, Institutions (Woff)], [Châtelet, Naturlehre].

Newton stated:

> Omnis enim Philosophiae difficultas in eo versari videtur, ut a Phaenomenis motuum investigemus vires Naturae, deinde ab his viribus demonstremus phaenomena reliqua. [Newton, Principia, Preface][23]

Euler introduced an alternative approach where the relation to geometry is not assumed. Therefore, the questions which previously emerged if geometrical curves were analyzed does not persist any longer. Euler based the calculus on the transfer of the rules, i.e. addition, subtraction, multiplication and division, valid for *finite* quantities to the operation with *infinitesimal* quantities. As a consequence, Euler considered infinitesimal quantities of different magnitude which are related to infinite quantities of different magnitude [Euler E387, § 84]. All these quantities are to be treated as *numbers* [Euler E101, §§ 1 to 10] and are allowed to appear in expressions composed of infinitesimal, finite and infinite quantities [Euler E212].

In the 20th century, the legacy of Newton and Leibniz had been reconsidered by those authors who did not confine the analysis to the traditionally preferred subjects like physics and metaphysics, but reconstructed the whole spectrum of activities of the great forerunners of modern contemporary science in the 17th and 18th centuries. The view at the history of science had been renewed

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[21] The most essential difference is due to the rejection of *inertia* or the *force of inertia* by Leibniz [Leibniz, Specimen, I (10)]. It may be concluded that the introduction of the variety of forces later introduced by Leibniz is a consequence of this rejection of inertia. Therefore, Leibniz could not escape from the necessity to accept indirectly the inertia as a general property of bodies.

[22] The rejection of perpetual motion follows from Leibniz's statement: "(...) nec plus minusve potentiae in effectu quam in causa contineatur" [Leibniz, Specimen, I (11)], there is neither more nor less potency in the effect than in the cause, i.e. the conclusion (not the assumption) is that the only remaining case follows as the equality of cause and effect or the conservation of the potency.

[23] "(...) for all the difficulty of philosophy seems to consist in this – from the phænomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phænomena; (...)." [Newton, Principia, Preface]

by Dijksterhuis [Dijksterhuis], Truesdell [Truesdell], Westfall [Westfall, Never], Jesseph [Jesseph, Berkeley], [Jesseph, Leibniz], Arthur [Arthur, Newton], [Leibniz Edition BBAW], [Newton Project] and related to the contemporary key problems in basic research in physics by Smolin [Smolin] and Wilczek. In recent papers, Wilczek stressed the essential difference between the original version of Newton's mechanics and the contemporary methods known from *quantum mechanics* [Wilczek 2004a], [Wilczek 2004b], [Wilczek 2005].

> When I was a student, the subject that gave me the most trouble was classical mechanics. That always struck me as peculiar, because I had no trouble learning more advanced subjects, which were supposed to be harder. Now I think I've figured it out. It was a case of culture shock. Coming from mathematics, I was expecting an algorithm. Instead I encountered something quite different – a sort of culture, in fact. Let me explain. (...) I won't belabour the point further. To anyone who reflects on it, it soon becomes clear that $F = ma$ by itself does not provide an algorithm for constructing the mechanics of the world. The equation is more like a common language, in which different useful insights about the mechanics of the world can be expressed. To put it another way, there is a whole culture involved in the interpretation of the symbols. When we learn mechanics, we have to see lots of worked examples to grasp properly what force really means. It is not just a matter of building up skill by practice; rather, we are imbibing a tacit culture of working assumptions. Failure to appreciate this is what got me in trouble. [Wilczek 2004a]

Expecting an algorithm, Wilczek formulated the problem of understanding from the same point of view as Euler who also intended to introduce an algorithm for the formulation and solution of mechanical problems. Euler mentioned the disadvantage of geometrical methods used by Newton and other authors. Although the reader may be convinced of the truth of the results, he cannot solve problems which differ only slightly in some details from the previously treated model.

In this book we will demonstrate that Euler discussed both the parts of Newton's theory, on the one hand, the absolute time and the absolute space and, on the other hand, the relative time and the relative space. Despite the final rejection of absolute motion, only the consideration of both parts of Newton's theory allows for a complete formulation of the space-time problem. The reason is that any physical theory can be considered as composed of *absolute* and *relative* quantities. However, the kind of such quantities has been considerably changed. Nowadays, instead of time or space the *velocity of light* is considered as independent of the state of motion of bodies, playing the role of an absolute quantity. Another prototype of such fundamental constants is *Planck's action parameter* as far as quantum theory is concerned. Newton mentioned that the relative time is "a measure of duration by means of motion". Therefore, the whole frame of concepts is built from time, space and motion. Euler continued the Newtonian tradition and discussed simultaneously the principles of mechanics for absolute motion and relative motion [Euler E015/016], [Euler E289].

In 19th century the development of the theory of relativity was heralded by Mach's criticism of Newton's concept of absolute space and time. Mach neither relied on Leibniz nor on Euler. On the contrary, he condemned Leibniz for his metaphysics and criticized Euler for his digression in writing the *Letters to a German Princess* [Mach, Mechanik, p. 433]. Only in the 20th century Reichenbach

[Reichenbach, Space and Time] acknowledged the merits of Leibniz in the foundation of a relational theory of time and space which was in contrast to Newton's approach. However, Euler's contribution did not attract the same attention. One of the reasons may be that Euler was considered in the first respect as a mathematician despite his tremendous contribution to physics. Chapters 10 and 11 of Euler's basic treatise *Anleitung zur Naturlehre* are devoted to that part of mechanics which is called nowadays relativistic mechanics [Euler E842]. The treatise is aimed towards investigation of *causes of changes* which are happened to bodies. Using the 18th century terminology we have to distinguish between *apparent* (*scheinbar* or *relative*) and *true (absolute)* motions.

The importance of Euler is caused by his unique ability to develop different branches of mathematics and likewise different sciences as mathematics and physics simultaneously. In the *Eulogy to Mr. Euler*, Condorcet [Condorcet, Eulogy] distinguished Euler's ability to make use of all branches of mathematics and all other sciences like physics, anatomy, chemistry and botany which was for him a source for discoveries nearly closed for every one else, but open to himself alone.

> Cette méthode d'embrasser ainsi toute les branches des mathématiques, d'avoir, pour ainsi dire, toujours présentes l'esprit toute les questions et toute les théories, était pour M. Euler une source de découverte fermée pour presque tous les autres, ouverte pour lui seul. [Condorcet, Eulogy]

As far as mathematics and mechanics is concerned Condorcet's statement may be readily confirmed by a comparison to Euler's mathematical writings until 1770 summarized in the comprehensive volumes: (i) *Methodus inveniendi lineas curvas*, 1744 [Euler E065], (ii) *Introductio in analysin infinitorum*, vol. 1 and 2, 1748 [Euler E101/102], (iii) *Institutiones calculi differentialis*, vol. 1, 1755 [Euler E212], (xi) *Institutionum calculi integralis* vol. 1, 1763 [Euler E342], vol. 2, 1763 and 1769 [Euler E366] and vol. 3, 1770 [Euler E385] and (iv) *Vollständige Anleitung zur Algebra*, vol. 1 and 2, 1770 [Euler E387/388].

In the end of the 18th century, the legacy of Euler strongly influenced the development. This impact is continued in the 19th century since one third of Euler's legacy had been only posthumously published.

In the end of the 19th century, the contributions of Euler to mathematics had been discussed and reconsidered after the foundation of the calculus by Weierstraß (compare the summary by Klein [Klein, Elementarmathematik]). Klein analyzed the relations between the calculus of finite differences and infinitesimals which played an essential role in the first half of the 18th century, mainly represented by the work of Taylor [Taylor, Methodus] published in 1715 and Euler papers written between 1727 and 1750 presented in the treatise *Institutiones* [Euler E212].[24] Although mathematicians were searching for the missing rigorous foundation of the calculus, physicists were accustomed to the operations with differentials introduced by Newton and

[24] "Ich möchte zuerst darauf hinweisen, daß das von Taylor zwischen Differenzen- und Differentialrechnung geknüpfte Band noch lange Zeit gehalten hat (...). Diese so naturgemäße Verbindung wurde erst durch die wiederholt erwähnten formalen Definitionen des Lagrangeschen *Derivationskalküls* aufgehoben." [Klein, Elementarmathematik, p. 253]

Leibniz and made successfully use of the calculus [Klein, Elementarmathematik],[25] [Bohlmann].

Currently, the tercentenary celebrations caused a new impact of Euler's legacy on the debate. However, people are mostly aware of special parts of Euler's work. Although, there is no doubt that even any of the parts of Euler's heritage is as influential as one can it imagine, the crucial determining factor comes from the interrelation between the parts an the correlation between the methods Euler applied and their diversity. Furthermore, in contrast to the legacy of Newton and Leibniz the edition of their papers is still progressing[26] and cannot be considered as finished, Euler's work is almost completely available for the "common reader" with the only exception of the letters [Euler Archive]. The study of Euler's work can be simultaneously founded and completed by the increasing deeper insight and hopefully entire reconstruction of the influence on the predecessors on Euler due to lively reception of the cornerstones of the debates running in the first half of the 18th century. As a result, Euler constructed between 1736 and 1756 a complete and consistent theory of classical mechanics based on the mechanical, mathematical and methodological principles developed by Descartes, Newton and Leibniz in the 17th century. In 1740, Châtelet presented a *methodologically* and *historically* designed summary of the controversial debates on the foundation of mechanics which completes advantageously Euler's *systematic* presentation in *Mechanica* [Euler, E015/016], *Anleitung* [Euler E842] and *Theoria* [Euler E289].[27] Although Châtelet did not obtain a satisfactory solution of the problems being a matter for debate, Châtelet prepared this matter for a solution by the same simultaneously performed two step procedure which had been also successfully applied by Euler, i.e. (a) to translate Newton's principles into the Leibnizian language and, (b) Leibniz's principles into the Newtonian language.

The analysis of Euler's theory will be mainly based on the treatises (i) *Mechanica*, 1736 [Euler E015/016], (ii) *Gedancken von den Elementen der Cörper*, 1746 [Euler E081], (iii) *Réflexions sur l'espace et le tems*, 1750 [Euler E149], (iv) *Découverte d'un nouveau principe de mécanique*, 1752 [Euler E177], (v) *Réflexions sur l'origine des forces*, 1752 [Euler E181], (vi) *Anleitung zur Naturlehre*, 1862 [Euler E842] and (vii) *Theoria motus corporum solidorum sive rigidorum*, 1765 [Euler E289], covering the three decades of Euler's work on mechanics from 1734 to 1765.

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[25] "Übrigens kommen diese naiven Betrachtungsweisen auch heute noch unwillkürlich zur Geltung, wenn man in der *mathematischen Physik*, der *Mechanik*, der *Differentialgeometrie* irgendeinen mathematischen Ansatz zustande bringen will. Sie alle wissen, daß sie überall da äußerst zweckmäßig ist. Freilich spottet der reine Mathematiker gern über eine solche Darstellung; als ich studierte, sagte man, daß für den Physiker das Differential ein Stück Messing sei, mit dem er wie mit seinen Apparaten umgehe." [Klein, Elementarmathematik, p. 227]

[26] Newton, The Newton Project [Newton, Project].

[27] Following Châtelet who addressed the Institutions to her son: "Ne cessez jamais, mon fils, de cultivar cette Science que vous avez apprise dès votre plus tendre jeunesse; (...)." [Châtelet, Institutions, II], Euler addressed 20 years later the *Lettres à une princesse d'Allemangne* [Euler E343], [Euler E344], [Euler E417] (written between 1760 and 1762).

Euler exemplified the promising strategy in all works which is far beyond a simple criticism or rejection of the predecessor's theory, but is always the result of a deep understanding and subsequent completion of the suppositions made by the authors. Although Euler excoriates essential parts of Leibnizian theory of monads, he credited Leibniz's methodology and mathematics, especially the representation of the calculus. Euler demonstrated that the theory of inherent forces is incompatible with basic principles of mechanics as far as the conservation of states is concerned. However, Euler performed a thorough analysis of theories called into question.[28]

In the beginning of 20th century, the rules quantum mechanics were proved to be essential different from those principles and interpretations of classical mechanics[29] which had been introduced Newton and Leibniz and their followers, respectively. The discrepancy to the classical interpretation was even more pronounced than in case of electrodynamics invented by Maxwell in 1875 whose notions, although the idea of *causal* connections represented by differential equations had been preserved, could not be modelled in terms of mechanics [Maxwell, Electromagnetism], [Maxwell, Treatise]. In the realm of atoms, Planck emphasized that it follows from the statistical interpretation of entropy, that causal connections known from Newtonian mechanics and represented by the calculus are completely missing [Planck, SelbstBiographie]. Nevertheless, the new development was based on the foundation which had been established in the 17th and 18th centuries and classical mechanics is indispensable also for the interpretations of new experimental findings [Bohr, Complementarity].

The book is organized as follows: In the Chaps. 1, 2 and 3 the mathematical and physical theories of Euler's predecessors Descartes, Newton and Leibniz are discussed particularly with regard to Euler's subsequent reinterpretation and synthesis. Euler's program for mechanics invented in 1736 will be discussed in Chap. 4. The progress Euler made will be demonstrated in Chaps. 4 and 5 for mechanics and mathematics, i.e. the foundation of mechanics and the foundation of the calculus, respectively. Especially, the difference between Newton's and Euler's foundation of mechanics will be demonstrated. In Chap. 5 Euler's invention of basic relations of the calculus will be analyzed from the point of view of contemporary non-standard analysis. In Chap. 6 it will be shown that this new approach is closely connected to

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[28] "16. Sollte man aber bei fleissiger Untersuchung diese Gründe nicht nur unrichtig befinden, sondern auch nach Verbesserung derselben, durch rechtmässige Schlüsse auf eine ganz andere Lehre von den Elementen der Körper gerathen: so würde dadurch nicht nur die Unrichtigkeit des Leibnizschen Lehrgebäudes von den Monaden deutlich dargethan, sondern auch an desselben Stelle die wahre Beschaffenheit der körperlichen Dinge erkannt werden." [Euler E081, II, § 16]

[29] Planck commented on the physical interpretation of the action parameter introduced by him: "So long as it could be regarded as infinitesimally small, i.e., dealing with higher energies and longer periods of time, everything was in perfect order. But in the general case difficulties would arise at one point or another, difficulties which became more noticeable as higher frequencies were taken into consideration. The failure of every attempt to bridge that obstacle soon made it evident that the elementary quantum of action plays a fundamental part in atomic physics and that its introduction opened up a new era in natural science, for it heralded the advent of something entirely unprecedented and was destined to remodel basically the physical outlook and thinking of man which, ever since Leibniz and Newton laid the ground work for infinitesimal calculus, were founded on the assumption that all causal interactions are continuous." [Condon]

Euler's foundation and application of the calculus which is based on Newton's and Leibniz's basic assumptions. The models of the world and the model for the bodies which Euler introduced to demonstrate basic law of mechanics will be analyzed. The extension and completion of the world models by the introduction of observers, called *Zuschauer*, will be discussed. Here, Euler's unification of Cartesian principles of motion which were developed and completed by Newton and Leibniz will be demonstrated for the distinctive features of Euler's transition from the Leibnizian *relational* approach of time and space to the *relativistic approach* in mechanics. It will be demonstrated that Euler modified both Newton's and Leibniz's assumptions introducing a new concept of the motion of bodies related rigorously to observers which had been later renewed by *Einstein*. It will be demonstrated that Euler developed the theory of mechanics in view of its consistency as well as its experimental confirmation. In Chap. 7 the development of mechanics by Helmholtz will be analyzed as an example for the 19th century approach based on the conservation law for energy. In Chap. 8 the discussion of the 20th century physics will be continued and completed. It will be demonstrated that the principles of Euler's mechanics can be also applied to discuss the foundation and interpretation of quantum mechanics given by Schrödinger.

The book is written primarily for physicists and mathematicians who are interest in the roots and the development of physical notions and theories. It may be also of interest for people who are involved in history of science, theory of science and philosophy since it naturally includes not only the ordinary theory of mechanics, but also those details of the development of mathematics and philosophy which are closely related to physics.

Contents

Chapter 1
The Predecessors: Descartes, Newton and Leibniz

The exceptional role of Descartes for the development of 17th century philosophy had been emphatically stressed in 19th century especially by philosophers[1] whereas his as essential contributions to mathematics and physics had not been separately treated, but preferentially in connection with philosophical foundation. This separation of the different parts of Descartes' work was not carried out in the 17th century nor completed in the 18th century.[2] On the contrary, Descartes' legacy was highly influential and stimulated the progress in the scientific community.[3] In the 18th century, there are essential developments of theories opposing Newton's theory.[4] Descartes' innovations achieved a sustained success in the work of Newton and Leibniz who referred to all constitutive parts of Descartes legacy and especially accentuated the correlation between metaphysics, mathematics and mechanics [Leibniz, Monadology, § 80]. However, there is an essential difference how Newton and Leibniz referred to Descartes. Leibniz openly admitted Descartes as most important

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[1] Wilhelm Windelband, *Geschichte der Philosophie*, Freiburg i. B., 1892 [Windelband]. Richard Falckenberg, *Geschichte der neerren Philosophie*, Leipzig 1892 [Falckenberg]. Karl Vorländer, *Geschichte der Philosophie*, Leipzig 1908 (First Edition 1902) [Vorländer].

[2] Following Descartes, the definition of the properties of bodies and, hence, the specific relation between bodies, had been explicitly distinguished from the properties of souls by Euler [Euler E842, § 49]. "Hier werden diejenigen Veränderungen mit Fleiss ausgeschlossen, welche unmittelbar von Gott oder einem Geiste hervorgebracht werden. Wenn wir also in der Welt nichts als Körper betrachten, so ist klar, dass ein jeder Körper so lange in seinem Zustande verbleiben muss, als sich von aussen keine Ursache ereignet, welche vermögend ist, in demselben eine Veränderung zu wirken. So lange aber die Körper von einander entfernt, so verhindert keiner, dass die Uebrigen nicht in ihrem Zustande, verharren könnten." [Euler E842, § 49] Obviously, Euler presupposed the whole set of Descartes's substances.

[3] Compare the summary of Châtelet from 1740 [Châtelet, Institutions] and Euler's comments on Descartes [Euler E842, §§ 9 and 35].

[4] Alternative models to Newton's theory of gravitation had been developed by Leibniz [Leibniz, Coelestium causis, 1698], Johann Bernoulli [Bernoulli J, La Nouvelle, 1734], Daniel Bernoulli [Bernoulli D, Recherches physiques, 1734] and Euler. Compare Nick K R, *Kontinentale Gegenmodelle zu Newton's Gravitationstheorie*, Dissertation, Frankfurt a. M., 2001. [http://deposit.ddb.de/cgibin/dokserv?idn=963471651&dok_var=d1&dok_ext=pdf.gz& filename=963471651.pdf.gz.] For recent analysis of the reception of Newton's theory see [http://www.ruhr-uni-bochum.de/wtundwg]. Euler discussed his theory of gravitation in *Anleitung zur Naturlehre* [Euler E842, Chaps. 12, 13, 14 and 15].

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predecessor whereas Newton only indirectly and reluctantly acknowledged the merits of Descartes [Westfall, Never, Chap. 7].

> 80. Descartes recognized that souls cannot impart any force to bodies, because there is always the same quantity of force in matter. Nevertheless he was of opinion that the soul could change the direction of bodies. But that is because in his time it was not known that there is a law of nature which affirms also the conservation of the same total direction in matter. Had Descartes noticed this he would have come upon my system of pre-established harmony. (Pref. [E. 477 a; G. vi. 44]; Theod. 22, 59, 60, 61, 63, 66, 345, 346 sqq., 354, 355.) [Leibniz, Monadology, § 80]

Descartes philosophy and methodology is known to be centred upon three concepts, (I) res infinita sive Deus, (II) res extensa sive corpus and (III) res cogitans sive mens and physics on three basic principles: (i) the conservation of state,[5] (ii) the continuation of motion in straight direction whose modification is only due to an external cause and (iii) the interaction of bodies in terms of their forces.[6] The invariance and permanence of the laws governing rest and motion of bodies are due the immutability of God. In the 19th century, Descartes system had been reduced to the mind-body dualism. However, in the 17th century, in the original non-dualistic version Descartes included an action of souls upon bodies changing its direction whereas the motion described by velocity is preserved. Descartes' theory had been analyzed and developed in this essential aspect by Malebranche, Spinoza and Leibniz. Leibniz removed the opposition between bodies and souls assuming a preestablished harmony. As a consequence, the opposition between rest and motion acquainted an important place and occupied a higher rank as in Descartes' mechanics where both the states of the bodies had been treated on an equal footing. The true inherent dualism of the res extensa becomes obvious when the res extensa is specified and decomposed into different bodies, at least two. Then, the dualism between *rest* and *motion* emerges.

Leibniz also was fully aware of the merits of Descartes. Having successfully treated Descartes' legacy as far as the *mind-body* problem is concerned by the theorem on the preestablished harmony between souls and bodies, Leibniz concentrates

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[5] "Prima lex naturae: quod unaquaeque res, quantum in se est, semper in eodem statu perseveret, sicque quod semel movetur, semper moveri pergat." [Descartes, Principia, Book II, § 37] "37. The first law of nature: that any object, in and of itself, always perseveres in the same state; and thus what is moved once always continues to be moved." [Descartes, Principles (Ross), Book II, § 37]

[6] "40. Third law: that a body, in colliding with another larger one, loses nothing of its motion; but, in colliding with a smaller one, loses as much as *it transfers to that one*. The third law of nature is this: where a body that is moved meets another, if it has less force [*vis*] to continue along a straight line than the other has to resist it, then it is deflected in another direction and, retaining its motion, loses only the determination of motion; if it has greater force, then it moves the other body with it and gives it as much of its motion as it loses." [Descartes, Principles (Ross), Book II, § 40] Here, Descartes *assigned a force to the uniformly moving body* thereby introducing an ambiguity in the theory of motion and forces which caused most of the confusion in the later debates on the measure of living forces (compare Chaps. 2, 4 and 7). Following Newton and Euler and referring to the 3rd Law, there is "neither less nor more force" of one of the bodies in the interaction of bodies, but always "forces equal in magnitude, but opposite in direction" since the action is always equal to the reaction.

on the relation between *rest* and *motion* as far as the motion of bodies and the phenomena are concerned. Leibniz reformulated Descartes' theory of the conservation of states in terms of a body - force problem. Finally in 1710, Leibniz summarized the state of art as a conjecture that even Descartes would have constructed the same correlation between the mind-body problem and the mechanical problem of the conservation of state.[7] Leibniz demonstrated the connection between the conservation law formerly treated as metaphysical principle [Leibniz, Specimen, I (11)] and the representation of motion in terms of geometry.

Newton treated the legacy of Descartes differently and introduced absolute time and space to *remove* Descartes' relativism as far as *relative motion of bodies* is concerned. However, in the 1st Law Newton [Newton, Principia, Axioms] preserved Descartes' basic assumptions as far as the bodies are described as things which are able to preserve their state represented by Descartes' operational statement "quantum in se est". By similar reasons, Newton invented the concept of *absolute space* to avoid the indeterminacy in the magnitude of velocity of a body.

Nowadays, this essential influence of Descartes on Newton's theory had been analyzed by Westfall [Westfall, Never]. However, Newton's relation to Descartes was essentially different from Leibniz's relation to Descartes. Newton also opposed the mathematical and the mechanical approaches of Descartes as far as the algebraic methods introduced beside and instead of geometry were concerned[8]. Analyzing Newton's manuscripts, Westfall concluded that Newton was not ready to confess the importance of Descartes' theory for himself and to acknowledge his merits. Contrary to Leibniz who explicitly and officially referred to Descartes, Newton even left out the name of his predecessor in unpublished manuscript *Matheseos universalis specimina* [Newton, Never].

> In chapter 4, Newton also repeated his attack on modern analysts, including Leibniz among them by implication. One fascinating sentence suggests the depth his revulsion against Descartes had reached. After expounding his fluxional method, he paused to reflect.
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> 'On these matter I pondered nineteen years ago, comparing the findings of .... and Hudde with each other.'

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[7] Later, Euler made use of the same interpretation of Descartes' legacy for the determination of the subject of mechanics. Following Descartes and Leibniz, Euler assumed that the world is made up of ghosts and bodies where the realm of bodies is as autonomous as the realm of ghosts as far the basic principles of action are concerned. Although all *influence* of ghosts on the motion of bodies is carefully excluded ("mit Fleiss ausgeschlossen") [Euler E842, § 49], the *relative motion* of bodies is the only motion which can be observed by ghosts [Euler E842, § 77] (compare Chaps. 4 and 6). Hence, there should be a correlation between "observers" and "bodies" which is different from any kind of interaction.

[8] "Men of recent times, eager to add to the discoveries of the Ancients, have united the arithmetic of variable with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output, but the advance is less of a blessing if you look at the complexity of the conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout way quantities which in geometry were designated by the drawing of a single line." [Newton, Math 4:421]

> The silence of the blank is deafening. Only one name – Descartes – could have belonged there. Newton could not longer bring himself even to acknowledge his debt. [Westfall, Never, p. 401][9]

Descartes established a relation between the basic concepts of mathematics and mechanics which was completed and generalized by Leibniz whereas Newton created an alternative concept in mathematics, i.e. the foundation of the calculus by time-dependent quantities called fluents and fluxions and mechanics mainly based on the geometrical methods of the ancients. Leibniz based the calculus on algorithms for differences and sums or on the analysis of curves where both the approaches are not related to time dependent quantities (compare Chap. 3).

Currently, Descartes is mainly present in the memory of people as philosopher and preferentially as the inventor of the mind-body dualism although his metaphysical system is centred upon three substances. Not surprisingly, the legacy of Leibniz is also preferentially comprehended and interpreted in terms of his contributions to metaphysics, philosophy and theology (compare the early criticism by Voltaire [Voltaire, Élémens] and the later version renewed by Mach [Mach, Mechanik]).[10] In the 19th century, Newton's writings on alchemy and theology[11] were widely ignored or masked by the overwhelming success of Newton's physics and mathematics. As a result, the constitutive components and their connections in Descartes' theory had been reduced and the ranking, i.e. the priority of philosophy had been preserved up to modern times. Descartes was preferentially considered as the founder of philosophical system of mind-body dualism. This reduced and selective perception was, however, observed neither in the 17th nor in the first half of 18th century where the state of art was represented by the opposition and battle between the Cartesian and Newtonian schools. The schools were represented by the adherents of Descartes and Newton who are living preferentially in France or in England. Moreover, following Newton, Descartes belongs to those "men of recent time" who were "eager to add

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[9] Quoted from *The Mathematical Papers of Isaac Newton*, Vol. 4, 1674–1684, edited by D. T. Whiteside, Vol. 4, p. 571. [http://www.cambridge.org/catalogue/searchResult.asp?ipcode=202022&sort=Y]

[10] "Bei Leibniz, dem Erfinder der besten Welt und der prästabilierten Harmonie, welche Erfindung in Voltaire's (...) Roman 'Candide' ihre gebührende Abfertigung gefunden hat, brauchen wir nicht zu verweilen" [Mach, Mechanik, p. 431]. Following Voltaire, Mach replaced the "best of all possible worlds" with the "best world" and interpreted the problem of choice of an object, the real world, based on an extreme principle as a problem of justification. "54. And this reason can be found only in the convenience [convenance], or in the degrees of perfection, that these worlds possess, since each possible thing has the right to aspire to existence in proportion to the amount of perfection it contains in germ." [Leibniz, Monadology, § 54]

[11] Only in the 20th century, Newton's legacy had been completely analyzed and acknowledged and, moreover, the importance of interconnection between mathematics, physics, theology and metaphysics had been analyzed (see e.g. [Keynes], [Truesdell], [Westfall, Never], [Guicciardini, Reading], [Guicciardini, Contextualism], [Snobelen]). "Recently the *Principia* has been the object of renewed interest among mathematicians and physicists. This technical interpretative work has remained somewhat detached from the busy and fruitful Newtonian industry run by historians of science. In this paper will advocate an approach to the study of the mathematical methods of Newton's *Principia* in which both conceptual and contextual aspects are taken into consideration." [Guicciardini, Contextualism]

to the discoveries of the Ancients, have united the arithmetic of variable with geometry" [Newton, Math 4:421]. Newton argued that the geometric method is "more elegant" than the algebraic calculus.[12]

> Indeed their method is more elegant by far the Cartesian one. For he achieved the result by an algebraic calculus which, when transposed into words (following the practice of the Ancients in their writings), would prove to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple proportions, judging that nothing written in different style was worthy to be read, and in consequence concealing the analysis by which they found their constructions. [Westfall, Never, p. 379] Newton's paper is entitled *Veterum loca solida restituta* [Newton, Math 4, 277].

Newton's aversion against Descartes' procedure in mathematics and physics was full of ambiguities since he was not only enforced and stimulated to go beyond Descartes, but also to invent alternative methods in the treatment of mechanical and mathematical problems. Hence, as a result, the representation of the fluents and fluxions in terms of *time dependent* quantities results from the exclusion of a purely algebraic approach. The breakthrough attained by the new method is compensated and even reduced by the restriction to accept exclusively time as the one and only one independent variable.[13] Already before Newton had published a short exposition of the *Method of Fluxions* in the *Principia* in 1687, the exceptional role of time was indirectly questioned by Leibniz who invented a new measure for forces called "force of motion" instead of the Cartesian measure of the "quantity of motion". Leibniz claimed that the Cartesian measure of the "quantity of motion" is not appropriate to be a measure of those forces which are generated by the motion of falling bodies of different masses traversing *different distances* in their motion [Leibniz, Brevis].[14] Leibniz's paper was the origin for the long lasting debate on the measure of *living forces* (compare Chaps. 2, 4 and 7).[15]

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[12] This argument had been later renewed in the debate about the priority in the invention of the calculus (compare Chap. 3). "(...) on the contrary, the Method of Fluxions, as used by Mr. Newton, has all the Advantages of the Differential, and some others. It is more elegant, because in his Calculus there is but one infinitely little Quantity represented by a Symbol, the Symbol *o*." [Newton, Account]

[13] Nevertheless, Newton presented the constraint resulting from this restriction as an advantage (compare previous footnote). Moreover, after Leibniz had published an appropriate *algebraic representation of differentiation and integration* in 1684 [Leibniz, Nova Methodus] which had been commonly accepted with the exclusion of the English [Euler E212], Newton vehemently defended his notation even thirty years later [Newton, Account]. Notwithstanding the question to whom the priority in invention of the calculus had to be allocated, Leibniz had the everlasting merit to have generalized the special relation between Newtonian "fluents" and "moments" as time dependent quantities to the general relation between variables of any kind and their differentials.

[14] "(...) inde factum est ut Cartesius, qui *vim motricem et quantitatem motus* pro re aequivalente habebat, pronunciaverit eandem quantitatem motus a Deo in mundo conservari." [Leibniz, Brevis]

[15] Finally, the *equivalence* of time and space as *independent* variables had been analytically and mechanically demonstrated by Euler who proved the validity of the relations (i) $m\ dv = K\ dt$ (Cartesian and Newtonian measure) and (ii) $m\ v\ dv = K\ ds$ (Leibnizian measure) between the change of velocity dv in dependence on the mass m, force K and time element dt and spatial element ds, respectively. [Euler E015/016, §§ 131–152] (compare Chaps. 2, 4 and 6). Euler also demonstrated that the distinction between the measures is not removed by this equivalence, but, on

After the similar developments of Newton and Leibniz as far as the invention of the calculus between 1665 and 1678 is concerned, it might be elucidating to compare also their later developments in the 1680's years. In the early period, both of them studied the works of Descartes and, later, both of them turned over to criticism of basic assumptions in Descartes' mathematics and mechanics. Newton studied at first the mathematical writings of Descartes before he read Euclid.[16] Similarly, Leibniz was at first in favour of Descartes' mechanics and the concept of inertia before he introduced the notion of forces by principles added to geometry[17] which Leibniz called metaphysical principles [Leibniz, Specimen, I (10)].[18] Following Westfall, Newton reread Descartes' works in the 1680's and found "errors in Descartes' *geometry*" whereas Leibniz reported on "errors in Descartes' *mechanics*" some year later [Leibniz, Brevis].

> What Newton had repeated to Pemberton were opinions that he first developed around 1680. (...) At much the same time, Newton reread Descartes' *Geometry*. Sixteen years earlier (...) it had introduced him to mathematics. (...) Probably in connection with this reading, he drew up a paper of 'Errors in Descartes' Geometry'. In the future, he never referred to his debt to Descartes. (...) Perhaps it was the vehemence of his revulsion that led him to undertake at this time a revised statement of his fluxional calculus which would place it on a solid geometrical foundation free of any taint of modern analysis. [Westfall, Never, pp. 378–380]

As it had been already shown above, Leibniz referred to Descartes differently and claimed that Descartes would come to similar conclusions in case of the relations between souls and bodies solely guided by the progress made in mechanics. In contrast to Newton, Leibniz accepted the *conservation of motion* which had been introduced by Descartes to exclude *perpetual motion*, but found a fault with the proposed measure in terms of the product of mass and velocity and replaced it with the product

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the contrary preserved and fully comprehended in its importance for mechanics [Euler E842, §§ 60–76].

[16] "If we judge by his manuscripts record from the years of silence, Newton's life during this period consisted primarily of theological and alchemical studies. (...) Mathematics (...) retained its capacity to excite him, and sometime in or near the year 1680, he devoted some attention to it. For the first time, he became a serious student of classical geometry. Years later, Henry Pemberton testified to Newton's high regard for the ancient geometers.

'Of their taste, and form of demonstrations Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclide with that attention, which so excellent a writer deserves.' [Pemberton] What Newton had repeated to Pemberton were opinions that he first developed around 1680." [Westfall, Never, pp. 377–78]

[17] "(...) quod in corpore praeter magnitudinem et impenetrabilitatem poni debeat aliquid, unde virium consideratio oriatur, cujus leges metaphysicas extensionis legibus addendo." [Leibniz, Specimen, I (11)]

[18] "Mihi adhuc juveni, et corporis naturam cum *Democrito* et hujus ea in re sectatoribus *Gassendo* et *Cartesio*, in sola massa inerte tunc constituenti, excidit Libellus *Hypothesos physicae* titulo, quo Theoriam motus pariter a systemate abstractam, et systemati concretam exposui (...)." [Leibniz, Specimen I (10)] The treatise was entitled *Theory motus abstracti* and *Theoria motus concreti* [Leibniz, Theoria Motus] (compare Chap. 2).

of mass and the square of velocity [Leibniz, Brevis] (compare Chaps. 2, 4 and 7). Although Leibniz completed the basic principles of mechanics by introducing the measure of motion by the product of mass and the square of velocity, the reformulation of the conservation of motion in terms of the conservation of moving force which had been later called "living force" caused much trouble and a long-lasting debate which had been only finished in the 18th century.

Leibniz demonstrated the connection between the conservation law formerly treated as metaphysical principle [Leibniz, Specimen, I (11)] and representation of motion in terms of geometry. The latter concepts are treated in spirit of Descartes' relational approach and is preserved without restrictions or reservations [Leibniz, Specimen, II (2)]. However, although Leibniz assumed that the relational approach in the theory of time and space is *necessary*[19] he claimed that it is not also *sufficient* since it is only adapted to the description of phenomena. Hence, similarly to Newton, Leibniz distinguished between phenomena and forces. The phenomena are relationally described in terms of space, time and geometry [Leibniz, Specimen, II (2)]. The theoretical component being sufficient is found in the concept of *forces* [Leibniz, Specimen, I (10), II (1)[20]].

In the first stage of development, Newton's theory published in 1687 was only acknowledged by the members of leading circles of the scientific community outside England. As an interesting detail, one can consider the interpretation and explanation of Newton's theory by people who were not directly involved in the creation of new mathematical and physical method, like Voltaire (1694–1778) [Voltaire, Élémens]. It is one of the merits of Voltaire to have initiated the break through for Newton's ideas at the continent by the *Lettres anglaises ou Lettres philosophiques* [Voltaire, Lettres] in 1734 and later the book on *Élémens de la philosophie de Newton* [Voltaire, Élémens] published in 1738. Voltaire wrote this book after returning from his visit in England between 1726 and 1729. In the subtitle of the *Élémens* Voltaire[21] stressed his intention to defend the *Newtonianism*, i.e. to defend Newton's theory against the Cartesianism and the adherent of the Leibnizian school. Although being in favour of Newton and attacking Leibniz, the frame of reference is always established by the basic concepts and principles introduced by

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[19] Newton treated the legacy of Descartes differently and introduced absolute time and space to *remove* Descartes' relativism as far as *relative motion of bodies* is concerned. However, in the 1st Law Newton preserved Descartes' basic assumptions as far as the bodies are described as things which are able to preserve their state represented by Descartes' operational statement "quantum in se est" [Descartes, Principia].

[20] "Natura corporis, imo substantiae in universum non satis cognita effecerat (...) ut insignes quidem philosophi nostri temporis, cum corporis notionem in sola extensione collocarent, ad Deum confugere cogerentur pro explicanda Unione inter Animam et Corpus, imo et communicatione corporum inter se. Nam fatendum est impossibile esse ut Extensio nuda solas involvens Geometricas notiones actionis passionisque sit capax; (...)." [Leibniz, Specimen II (1)].

[21] This comparative approach is expressed in the title of the Amsterdam edition of voltaire's book: *La métaphysique de Neuton ou parallèle des sentimens de Neuton et de Leibnitz* [Voltaire, Newton]. Voltaire discussed the difference between Cartesianism in France and Newtonianism in England already in 1734 [Voltaire, Lettres, Lettre XIV]. The chapter is entitled *On Descartes and Sir Isaac Newton* [http://www.bartleby.com/34/2/].

the great and celebrated predecessors Newton *and* Leibniz. In this time, the same approach had been chosen by the younger contemporaries of Voltaire, especially by Euler and Madame du Châtelet (1706–1749) [Châtelet, Institutions] who were in the same age as starting their scientific careers in the post-Newtonian period of development of mathematics and mechanics. The post-Newtonian period was simultaneously also a post-Leibnizian period superimposed by the reception and the influence of the legacy of Descartes.

Euler analyzed carefully the basic concepts and methods introduced by Descartes, Newton and Leibniz. A comprehensive review had been given in the *Anleitung zur Naturlehre* [Euler E842] (compare Chaps. 4 and 6) and later in the *Lettres à une Princesse d'Allemagne* [Euler E343, E344, E417] after 1768. In all these treatises by Voltaire, Châtelet and Euler, the later 19th century interpretations and modifications of the Cartesian, Newtonian and Leibnizian theories were not present and did not distort the perception and understanding of the original version. Later, a special interpretation became dominant and the alternatives had been suppressed for a long time and only recovered with a certain time delay. Hence, the younger generation of scientist had not only to tackle the legacy of Descartes, but also the legacy of Newton and Leibniz who went considerable beyond the frame which had been established by Descartes.

Currently, we can make use of their work to reconstruct the breakthrough in the thinking and the methods invented by Descartes, Newton and Leibniz. All the scholars could not really escape from an analysis of the alternatives presented by their predecessors. The only alternatives were to become (i) either an adherent of one of the schools, either the Cartesian or the Leibnizian or either the Newtionian or the Leibnizian, (ii) or to take the chance to invent a synthesis (iii) or the construct an approach beyond the frame known from legacy. In latter cases, there was a need to search not only for the differences, but preferentially for the common basis. Then, deep rooted common presuppositions of the controversially discussed representations of apparently contradictory theories may be discovered and, moreover, it was possible to overcome the seemingly incompatible features being accentuated by the inventors.

Not surprisingly, the followers of the founders of new mechanics and mathematics proceeded differently. Some of them reconsidered the *whole legacy* of their predecessors by reducing the commonly established prejudices whereas others accentuated the discrepancies and enhanced the quarrels to display a supremacy which really could not be confirmed.[22] This procedure will be demonstrated by the controversy about the Newtonian and the Leibnizian versions of (i) the foundation of mechanics and the concepts of time, space and motion and (ii) the dispute about the

[22] Voltaire took an intermediate position in analyzing the 17th century theories, but was finally in favour of Newton and criticized Leibniz preferentially by philosophical reasons. Châtelet, on the contrary, came to different conclusion, probably due to the different perception caused by the mathematical and physical parts of Leibniz's legacy. Voltaire and Châtelet represent the split into a metaphysically and a methodologically motivated criticism, respectively. This development had been later continued in the end of the 18th century and the mid of the 19th century by Kant and Mach, respectively.

invention of the calculus. Reading the works of Euler[23] and Châtelet,[24] we may get an interesting insight in the fascinating process of comprehension of the key problems in the beginning of the 18th century and their reformulation within a new context which finally reduced the old controversies to mere differences of the representation of the same subject in two equivalent languages [Kästner, Anfangsgründe].[25]

Leibniz's pioneering work on logic and scientific methodology had been only appreciated in 19th century. In 18th century and later Leibniz's merits were masked and suppressed by the misleading criticism of his metaphysics by Voltaire and Mach. However, with a certain delay, the Leibnizian contribution had been finally mentioned and highly acknowledged. So, Couturat and Russell referred explicitly to Leibniz, and later, in 1920 Reichenbach appreciated Leibniz's relational theory of space and time. Leibniz's methodological principles can be considered as a reliable basis for the understanding and the development of mechanics. The *principle of continuity* and the role of *order* and *contingency* as leading and superordinate principles are of greater importance only as being known from their one-sided and restricted metaphysical interpretation.

1.1 The Reception of the Legacy of Descartes, Newton and Leibniz

> VI. Les systèmes de Descartes & de Newton partagent aujourd'hui le monde pensant, ainsi il est nécessaire que vous connaissiez l'un & l'autre. [Châtelet, Institutions, VI]

The tercentenary celebrations of the births of Emilie du Châtelet in 2006 and Leonhard Euler in 2007 are most welcome to reconstruct the development of mechanics during the post-Newtonian period in the first half of 18th century [Hagengruber, 2007], [Hagengruber, 2008], [Hagengruber, Naturlehre], [Hagengruber, Metaphysik]. The reception of the legacy of Descartes, Newton and

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[23] Euler constructed between 1736 and 1756 a complete and consistent theory of classical mechanics based on the mechanical, mathematical and methodological principles developed by Descartes, Newton and Leibniz in the 17th century (see Chaps. 4, 5, 6 and 7). For the criticism of Leibniz's theory compare *Gedancken* [Euler E081, II, §§ 16 and 17].

[24] In 1740, Châtelet [Châtelet, Institutions] presented a *methodologically* and *historically* based summary of the controversial debates on the foundation of mechanics entitled *Institutions de physique* which completes advantageously Euler's systematic presentation. Although Châtelet did not obtain a satisfactory solution of the problems being a matter for debate, Châtelet prepared this matter for a solution by the same simultaneously performed two step procedure which had been also successfully applied by Euler, i.e. (a) the translation of Newton's principles into the Leibnizian language and, (b) the translation of Leibniz's principles into the Newtonian language. Although being an adherent of Leibniz's methodology, Châtelet was equally in favour of Newton's *Principia* and translated Newton's *Principia* into French [Newton, Principia (Châtelet)].

[25] "Dieser Streit ist also, ihm den rechten Nahmen aus der griechische Grundsprache zu geben, eine Logomachie gewesen; doch ist selbst ein solcher Wortstreit, in der Mathematik lehrreicher gewesen; als Wortstreite in anderen Theilen der Gelehrsamkeit zu seyn pflegen. Denn die vielfältigen Untersuchungen, Auflösungen und Aufgaben u.d.g. die er veranlaßt hat, haben doch unsere Kenntniß wirklich erweitert." [Kästner, Anfangsgründe, § 202]

Leibniz will be reconstructed in terms of the 18th century approaches of those people like Voltaire, Euler and Châtelet who were, at first, by natural reasons not familiar with the 19th and 20th century physics and, at second, also not familiar with the 19th and 20th interpretations of the metaphysical systems of Descartes and Leibniz.[26] Obviously, this approach may bring forth restrictions and results in a reduction of the information we can obtain about the historical development since many, not to say most of the works of Leibniz and essential parts of the works of Newton have been only published in 19th and 20th century. Leibniz's considerations on logics and Newton's interest in alchemy and theology were unknown or not counted for a long period.[27] However, the disadvantage of such procedure is compensated or even overcompensated caused by the change to a *historical frame of reference*. The advantage is that we can make use of the original interpretation which had been given by people who were not only involved in the development as neutral observers looking from outside or a posteriori at the scene, but who were forced to do the next step in the track of their great predecessors. In contrast to us, who already know the answers, they have not only to *answer* the questions, but they have also and in first respect to *formulate* the questions for the answers we know today. Therefore, we are in the comfortable position that we cannot only *analyze* different answers, but we can also analyze the *origin* of different answers. This procedure may be also useful for a critical reading and reconsideration of the questions and answers which result from the contemporary development.

There is, additionally, a simple reason why this procedure seems to be appropriate. The long-lasting discussion of the basic problems of the foundation of classical mechanics had not been finished with the invention of quantum mechanics, but, after a long period where some questions had been only postponed,[28] rather renewed [Wilczek 2004a].[29]

[26] The 18th century developments of mathematics had already been highly acknowledged in the end of the 19th century. "Gemeinhin verbindet man mit dem Begriffe der Mathematik schlechtweg die Idee eines streng logisch gegliederten auf sich selbst ruhenden Systems, (...). Indes ist der Geist, aus dem die moderne Mathematik geboren wurde, ein ganz anderer. Von der Naturbeobachtung ausgehend, auf Naturerklärung gerichtet, hat er ein philosophisches Prinzip, das *Prinzip der Stetigkeit*, an die Spitze gestellt. So ist es bei den großen Bahnbrechern bei *Newton* und *Leibniz*, so ist das ganze 18. Jahrhundert hindurch, welches für die Entwicklung der Mathematik recht eigentlich ein Jahrhundert der Entdeckungen gewesen ist." [Klein, Arithmetization]

[27] "If we judge by his manuscript record from the years of silence, Newton's life during this period consisted primarily of theological and alchemical study. (...) There was no point during these years when Newton divorced himself entirely from mathematics. (...) Mathematics was never solely a duty to Newton. It retained its capacity to excite him, and sometimes in or near the year 1680, he devoted some attention to it." [Westfall, Never, p. 377]

[28] In 1787, Gehler summarized: "Kraft. Ein allgemeiner Name alles dessen, was Bewegung hervorzubringen, zu ändern oder zu hindern strebt. Daß diese Ursachen der Bewegung in der tiefsten Dunkelheit verborgen liegen, und ihr erster Ursprung außer der Körperwelt gesucht werden müsse, ist schon bey dem Worte: Bewegung erinnert worden. Da indeß jede Aenderung des Zustands einen Grund, mithin auch jede Entstehung und Veränderung der Bewegung eine Ursache voraussetzt, so behelfen wir uns mit dem Worte: Kraft, um dadurch alle diese Ursachen zu bezeichnen, die wir so oft nennen müssen, obgleich ihre Natur ein unerforschliches Geheimniß bleibt." [Gehler, Kraft]

[29] See the titles of a series of papers by Wilczek. F. Wilczek, *Whence the Force of* $F = ma$? I: Culture Shock, Physics Today, Vol. 57, October 2004. *Whence the Force of* $F = ma$?

In the 20th century, similar to the impressive discoveries of Leibniz's contributions to mathematics and logics by Couturat [Couturat, Leibniz] and Russell [Russell, Western] in 19th century, we find an as impressive development in the foundation of the calculus by the invention of non-standard analysis due to Robinson [Robinson], Laugwitz and Schmieden [Schmieden Laugwitz] in 20th century.

In the 18th century, the progress in development was dominated by those who decided to reconsider the legacy of their predecessors without prejudice. Châtelet accentuated this common feature in the treatment of Descartes' and Newton's legacy:

> VI. Les systèmes de Descartes & de Newton partagent aujourd'hui le monde pensant, ainsi il est nécessaire que vous connaissiez l'un & l'autre. (...) VII. Gardez-vous, mon fils, quelque parti que vous prenez dans cette dispute des Philosophes, de l'entêtement inévitable dans lequel l'esprit de parti entraîne: (...) mais il est ridicule en Physique, la recherche de la vérité est la seule chose dans laquelle l'amour der votre pais ne doit point prévaloir, & c'est assûrément bien mal-à-propos qu'on a fait une espéce d'affair nationale des opinions de Newton, & de Descartes. [Châtelet, Institutions, VI and VII]

Between 1734 and 1756 Euler, Maupertuis, Voltaire, Châtelet and d'Alembert published important writings on all controversial topics treated before by Newton and Leibniz. In 1734, Euler published a two volume treatise entitled *Mechanica sive motus scientia analytice exposita* [Euler E015/016]. In 1738, as a result of his stay in England and the collaboration with Châtelet, Voltaire published the *Élémens de la philosophie de Newton* [Voltaire, Élémens]. Châtelet's *Institutions de physique* [Châtelet, Institutions] followed in 1740 and, finally, in 1743, d'Alembert's *Traité de méchanique* [d'Alembert, Traité] issued. D'Alembert commented on Daniel Bernoulli and Euler. Later, in 1756, Maupertuis' analyzing the principles of Descartes, Newton, Leibniz and Huygens commented also on Euler [Maupertuis, Examen] and considered Euler's principle[30] as a novelty.[31]

Châtelet's *Institutions* may be considered as a handbook for the above mentioned translation of basic concepts translation including explicitly almost all translations rules for the mechanical concepts, especially for the different kinds of forces. Analyzing the same subjects, Euler proceeded in similar way, but adjoined, moreover, mathematical rules concerning the relation between infinitesimal and finite quantities. We may guess that also all other authors in the 18th century who participated in the development of mechanics were confronted with the same problems. In view of the completeness, Châtelet's *Institutions* may be compared to Maupertuis' *Examen* [Maupertuis, Examen] written in 1756. Euler's reviews are distributed over all

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II: Rationalization, Physics Today, Vol. 57, December 2004. *Whence the Force of* $F = ma$? III: Cultural Diversity, Physics Today, Vol. 58, July 2005. Here, based on the well-established principles and foundation of quantum mechanics, the "inverse cultural shock" appeared to encounter classical Newtonian mechanics which was observed one century before as classical mechanics encountered theory of relativity or quantum mechanics.

[30] Euler emphazied the importance of the principle. The paper is entitled *Découverte d'un nouveau principe de Mécanique* and the principle is thought to be valid for whole mechanics since the laws for the motion of extended rigid bodies had been derived: "Principe général et fondamental de toute la mécanique." [Euler E177, § 20]

[31] Therefore, at that time no commonly accepted basic principle of mechanics was available although Newton invented a new approach.

his writings. In case of *relative motion* it may be extracted from *Mechanica* (1736), *Anleitung* ($\sim$1750, published 1862) [Euler E842], *Theoria* (1765) [Euler E289] and *Lettres* (1765–1768) [Euler E343/344]. In case of the calculus it may be taken from *Mechanica* and *Institutiones* (1755) [Euler E212]. Therefore, Châtelet had only the chance to read and to comment on the *Mechanica*.

The challenge to the scientific community was to understand the correlation between the theories of Descartes, Newton and Leibniz. The post-Newtonian period is also a post-Leibnizian period. Newton's mechanics had been always discussed in relation to Leibniz's theory of time and space and the measure of living forces. Independently of the different points of view introduced by different authors, the heritage of the great predecessors is always present in their writings.[32] As it had been discussed above, a representative review of the state of art and the controversies might be obtained from Voltaire's *Letters* [Voltaire, Lettres], *Élémens* [Voltaire, Elements] and Châtelet's *Institutions* [Châtelet, Institutions] which are only surpassed by Euler's reflections and comments on the debates outlined in the *Lettres à une Princesse d'Allemagne* [Euler E343], [Euler, E344], [Euler E417] written between 1760 and 1762 and published after 1768.

1.1.1 Newton Versus Leibniz: Voltaire

The contemporaries of Newton and Leibniz were challenged by the opposition and contradictions between the models invented by Descartes, Newton and Leibniz and their adherents.[33] Voltaire was very proud of his criticism of Leibniz's idea of the best of all possible worlds [Voltaire, Candide].[34] Probably, Voltaire might be disappointed about Russell's interpretation that Leibniz had adapted the presentation

[32] The end of the era was due to Planck only in 1900 (compare Chap. 8). In 1945, Planck summarized: "The failure of every attempt to bridge that obstacle soon made it evident that the elementary quantum of action plays a fundamental part in atomic physics and that its introduction opened up a new era in natural science, for it heralded the advent of something entirely unprecedented and was destined to remodel basically the physical outlook and thinking of man which, ever since Leibniz and Newton laid the ground work for infinitesimal calculus, were founded on the assumption that all causal interactions are continuous." [Condon]

[33] "A FRENCHMAN who arrives in London, will find philosophy, like everything else, very much changed there. He had left the world a plenum, and he now finds it a vacuum. At Paris the universe is seen composed of vortices of subtile matter; but nothing like it is seen in London. In France, it is the pressure of the moon that causes the tides; but in England it is the sea that gravitates towards the moon; so that when you think that the moon should make it flood with us, those gentlemen fancy it should be ebb, which very unluckily cannot be proved. For to be able to do this, it is necessary the moon and the tides should have been inquired into at the very instant of the creation." [Voltaire, Lettres, Lettre XIV]

[34] In the 19th century, Voltaire's criticism of Leibniz's theory had been acknowledged by Mach [Mach, Mechanik] who was also in favour of Newton. It was not mentioned that Leibniz presented a special version of *extremal principles* whose mechanical version was due to Maupertuis [Maupertuis, Repos], [Maupertuis, Mouvement] and Euler [Euler E065] and, later due to Lagrange [Lagrange, Mécanique] (variational approach) and Hamilton [Hamilton 1], [Hamilton 2].

of his philosophy to the taste of the public [Russell, Western Philosophy].[35] Hence, Voltaire had to pay a price for his delicious and ironic criticism. He had to make a caricature[36] of Leibniz's theory and introduced assumptions which are never found in the original writings, e.g. by applying the Leibnizian principle to a part of world instead of the world as a whole. Voltaire showed the "best of all possible princesses" or the "best of all castles" [Voltaire, Candide], i.e. items which cannot appear by principle reasons in Leibniz's theory where only the whole, but not also its parts are governed by extremal principles (compare [Leibniz, Monadology, § 54]). The step-by-step recovery of the original version of Leibniz's methodology had been initiated by Couturat (compare Russell's comment [Russell, Western, Chap. Leibniz]). Therefore, in the post-Newtonian period, each of the scholars had to solve the problem to give an adequate representation of Newton's and Leibniz's theories from incompletely or only fragmentarily published legacy.

Euler published his program for mechanics in 1736 [Euler E015/016], d'Alembert published the *Traité* in 1743 [d'Alembert, Traité] where he is also developing a method to tackle a new class of problems. Furthermore, after his stay in England, Voltaire published his view at Newton in 1738 [Voltaire, Élémens] and Châtelet published the *Institutions* [Châtelet, Institutions] in 1740 (second edition already in 1743). Maupertuis invented the principle of least action [Maupertuis, Repos], [Maupertuis, Accord]. The authors took notice from each other in their correspondence, but also by personal communication. In general, they treated the same problems, however, with different methods.

Furthermore, the scientific community was involved in the long-lasting debate on the true measure of *living forces*[37] and, as a consequence, split into different schools mainly localized in France, England and Germany.

The first attempts to reconsider the legacy were made by Voltaire and d'Alembert who invented a new procedure predominantly fixed upon the exclusion

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[35] "Apology is due to the specialists on various schools and individual philosophers. With the possible exception of Leibniz, every philosopher of whom I treat is better known to some others than to me. (...) I come now to Leibniz's esoteric philosophy, in which we find reasons for much that seems arbitrary or fantastic in his popular expositions, as well as an interpretation of his doctrines which, if it had become generally known, would have made them much less acceptable. It is a remarkable fact that he so imposed upon subsequent students of philosophy that most of the editors who published selections from the immense mass of his manuscripts preferred what supported the received interpretation of his system, and rejected as unimportant asays which prove him to have been a far more profound thinker than he wished to be thought. Most of the texts upon which we must rely for an understanding of his esoteric doctrine were first published in 1901 or 1903, in two works by Louis Couturat. One of these was even headed by Leibniz with the remark: 'Here I have made enormous progress.' But in spite of this, no editor thought it worth printing until Leibniz had been dead for nearly two centuries." [Russell, Western, Chap. Leibniz]

[36] Responding to Euler's *Mechanics*, Lichtenberg similarly got on with the model of mass point Euler had introduced in the theory [Lichtenberg, Sudelbücher].

[37] The origin of the debate can be traced back to Leibniz's comment on the "errors of Descartes" [Leibniz, Brevis] and Newton's *Principia* [Newton, Principia], published in 1686 and 1687, respectively, where independently of each other different representation of the relation between motion and forces were introduced (compare Sect. 4.4). The challenge to the scientific community to understand the correlation between the different approaches may be illustrated by comparison to a later as fundamental event, the creation of quantum mechanics by Heisenberg and Schrödinger in 1925 and 1926, respectively, by seemingly completely different methods. In contrast to the long lasting debate on the relation between dead and living forces, the equivalence was readily demonstrated by Schrödinger in 1926 [Schrödinger, Heisenberg].

of metaphysical assumptions and implications. Nevertheless, Voltaire considered Newton and Leibniz simultaneously, but the preference is obvious as it follows from the title of the treatise *Élémens de la philosophie de Newton* published in 1738. The Amsterdam edition is entitled *La métaphysique de Neuton ou parallèle des sentimens de Neuton et de Leibniz*.

Although Voltaire preferred Newton he did not fail to compare Newton to Leibniz and Leibniz to Newton in detail assuming his own methodological frame of reference founded in the enlightenment. Châtelet proceeded similarly, but surpassed Voltaire in *translating* Newton into the Leibnizian and Leibniz into the Newtoninan language. In contrast to Voltaire who stressed the *differences*, Châtelet intended to seek without prejudice[38] for *common* principles existing in Descartes, Newton and Leibniz, but might be hidden at a first glance.[39] The creation of the calculus may serve as a prototype of these *common basic principles* which are only expressed alternatively in outward different languages seemingly being opposed to each other.[40] From Châtelets *Institutions* we can learn to read the texts without making use of the knowledge of later interpretations.

1.1.2 Newton and Leibniz: Châtelet

> XII. Les idées de M. de Leibnits sur la Métaphysique, sont encore peu connues en France, mais elles méritent assûrément de l'être. [Châtelet, Institutions, XII]

Châtelet's *Institutions* [Châtelet, Institutions] and Maupertuis' *Examen* [Maupertuis, Examen] may be considered as a summary of the state of art in mechanics and metaphysics in 1740 and 1756, respectively. As Châtelet, Maupertuis analyzed the principles of Descartes, Newton, Leibniz, Huygens and Euler. Châtelet referred to Descartes' principles (the definition of the body as *res extensa*), to Leibniz's distinction between active and passive forces and to Newton's distinction between the "force of inertia" and the "moving force" which can be considered as an alternative version of Leibniz's treatment.

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[38] "Und es ist in der That ungereimt, daß man aus Cartesens und Newtons Meynungen eine Art von Nationenhändeln gemachet hat." [Châtelet, Naturlehre, VI]

[39] "The consequence is that there are two systems of philosophy which may be regarded as representing Leibniz: one, which he proclaimed, was optimistic, orthodox, fantastic, and shallow; the other, which has been slowly unearthed from his manuscripts by fairly recent editors, was profound, coherent, largely Spinozistic, and amazingly logical. It was the popular Leibniz who invented the doctrine that this is the best of all possible worlds (to which F. H. Bradley added the sardonic comment 'and everything in it is a necessary evil'); it was this Leibniz whom Voltaire caricatured as Doctor Pangloss. It would be unhistorical to ignore this Leibniz, but the other is of far greater philosophical importance." [Russell Western, Chap. XI] However, the "complete unpublished" Leibniz is always present also in the writimgs of the "published" Leibniz.

[40] As it will be demonstrated, the same internal similarity in the basic assumptions of Newton's and Leibniz's mechanics is found as far as the notions of time, space, motion and forces are concerned.

Châtelet translated Newton into Leibniz and Leibniz into Newton (compare the translation of the *Principia* into French) modifying, but preserving mainly the terminology [Suisky 2008]. Châtelet was not hindered, but rather stimulated by the controversies to search for common basis of both theories. Obviously, both authors stressed and overrepresented the differences in their approaches, but screened carefully and consciously the common basis. This complicated, but nevertheless very interesting relation and relationship may be demonstrated in case of the invention of the calculus. After a long-lasting debate and numerous subsequent investigations of the details of the debate summarized by the title of the book *Priority and equivalence* [Meli], the only conclusion can be that only Leibniz was able to understand immediately Newton's cryptography like, in the opposite case, only Newton was able to acknowledge immediately the progress Leibniz made. Therefore, Newton and Leibniz *agreed* almost perfectly in the meaning and weighting of the *basic* questions and problems as far as the legacy of *Archimedes, Galileo* and *Descartes* was concerned, but represented their developments and steps beyond their predecessors within a frame of notions and in languages which turned out to be quite *differently* from each other. Euler discovered this common basis generated by Newton and Leibniz.

The contemporary view back at this development in the 18th century was mainly formed in the 19th century and in the first half of the 20th century. It will be exemplified for the controversy on the "true measure of living forces". Following Mach's analysis, Newton is considered as physicist whereas Descartes and Leibniz are considered as adherents and creators of metaphysical systems [Mach, Mechanik]. The advent of this procedure can be traced back to Voltaire who criticized the "mania of systems" [Voltaire, Élémens].

However, Châtelet developed another scheme of reference and considered the contributions of Newton and Leibniz to mechanics on an equal footing from the point of view of a translator. Châtelet translated Newton's *Principia* after writing the *Institutions*. Therefore, it is not appropriate to read Châtelet's *Institutions* from an a posteriori position which was commonly accepted only after 1740, but from the point of view of those people who were involved in reading Newton *before* his theory was commonly accepted. It should be mentioned that Voltaire published his treatise *Élémens* which paved the way for an understanding of Newton's theory in France only in 1738.

A reliable source to get insight in the ranking between the predecessors and contemporaries is Maupertuis' *Examen* [Maupertuis, Examen] published in 1756. Maupertuis analyzed the *mechanical* principles of Descartes, Newton, Leibniz, Huygens and, finally also those introduced by Euler. Comparing different approaches and principles to each other an appropriate terminology is necessary where the common features are highlighted and the differences become visible.

Voltaire modified the reconstruction in favour of Newton, whereas Wolff proceeded in the other way and modified the reconstruction in favour of Leibniz. However, despite the obvious differences in their representation, a *hidden conformity* in the basic assumptions of Newton and Leibniz can be detected, e.g. those which are related to time and space. The ranking between time and space found in the title of Reichenbach's book on *The philosophy of space and time* [Reichenbach, Space and Time]. These concepts appear in Newton's *Principia* and Leibniz's *Initia* in an inverted order, time at fist and space at second, which demonstrates the difference in the weighting in 17th and 20th centuries, however, it reveals also a unexpected agreement between the *Principia* [Newton, Principia] and the *Initia* [Leibniz, Initia]. Despite their controversy on the definition of time and space Newton and Leibniz agreed almost perfectly in the *order* of introduction of the concepts of time, space and motion and, furthermore, as far as the relation between geometry and motion (compare Chap. 2).

This order had been only called into question by Euler who based his considerations on the axiom of the uniform motion of bodies [Euler E015/016], [Euler E149] (compare Chaps. 4 and 6). Therefore, the distinction between basic quantities is now made predominantly between absolute and relative motions rather than between absolute and relative time [Euler E015/016]. Euler's approach permits a new consideration of the whole problem since it contains implicitly the question whether the motion, described by the *velocity* of the body, is limited in its magnitude. Leibniz considered the rotation of a disk [Leibniz, De ipsa] and concluded that the velocity can be increased without limitation by the increase of the distance from the center of disk.

1.1.3 Descartes, Newton and Leibniz: Euler and Châtelet

Châtelet modified the Newton-Leibnizian order in favor of Descartes. The chapter on the space V. *De l'Espace* precedes the chapter on time, VI. *Du Tems* [Châtelet, Institutions]. As a consequence, the basic property of the bodies is considered to be the *extension* [Châtelet, Institutions, Chap. VIII][41] or, the assumption of the basic properties of the bodies results necessarily in an special order of introduction of time and space.

Euler surpassed all his contemporaries including Châtelet, but built from the very beginning a new foundation using the basic components of Descartes', Newton's and Leibniz's procedure in mechanics[42] mathematics[43] and methodology.[44] The title of his comprehensive treatise on mechanics represents simultaneously his program for mechanics: *Mechanica sive motus scientia analytice exposita*, published in 1736. Euler introduced the following essential topics which make his mechanics different from the theories of his predecessors, (i) the rigorous statement on the

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[41] Motion and rest are considered in Chap. XI *De Mouvement, & du Repos en général, & du Mouvement simple* [Châtelet, Institutions].

[42] These are the preservation of states, the preservation of momentum and the relations between different types of forces (compare Chap. 4).

[43] This concerns Euler's reconstruction of Leibniz's arithmetic foundation of the "calculus of differences and sums" [Leibniz, Elementa nova] (compare Chap. 5).

[44] These are the principles of contradiction and sufficient reason, the distinction between necessary and contingent truths, the logical consistency of mechanical theories, the need and the creation of special language for science.

priority of relative motion,[45] combined with the introduction of an *observer*, called or *spectator* [Euler E015/016 § 97 and §§ 7 and 80] in the preceding paragraph of his program for mechanics [Euler E015/016, § 98], comprehensively elaborated in the *Anleitung* where the observer is called *Zuschauer* [Euler E842, §§ 77–83] and maintained in the *Theoria* [Euler E289, §§ 1–11] (compare Chap. 6), (ii) the introduction of more than one observer[46] who compared the results of their observations, which results in the confirmation of (iii) the invariance of the equation of motion,[47] (iv) the explanation of the origin of forces[48] (compare Chap. 6) and (v) the harmony between mathematics and mechanics resulting from Euler's procedure to coordinate his the progress mathematics with his progress in physics. Euler transferred essential methodological rules from mathematics into mechanics and from mechanics into mathematics.

> Cette méthode d'embrasser ainsi toute les branches des mathématiques, d'avoir, pour ainsi dire, toujours présentes l'esprit toute les questions et toute les théories, était pour M. Euler une source de découverte fermée pour presque tous les autres, ouverte pour lui seul. [Condorcet, Eulogy]

Although the *Anleitung* may be considered as an almost complete representation of all mathematical and mechanical principles for the foundation of a relativistic mechanics, it has not been mentioned in contemporary treatments after the publication in 1862. The decisive impact was due Mach [Mach, Mechanik] in 1883.

1.2 The Common Basis: Descartes on Motion of Bodies

Descartes claimed that *extension*, rest and motion are the basic concepts of mechanics. Following Descartes a reliable theory of matter and motion is necessarily based on a *dualistic* basis, the dualism between rest and motion.[49] Although Descartes

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[45] Euler rejected absolute motion which had been considered by Newton. This difference to Newton is not acknowledged, on the contrary, it is often said that Euler is an adherent of absolute motion. "This is clear from the above passage, where Kant deliberately refrains from endorsing the Newtonian conception – adopted by Euler – of *absolute motion*." [Friedman]. Euler discussed and criticized Leibniz's relational theory of space and times in his paper *Réflexions sur l'espace et le tems* [Euler E149].

[46] Usually, the introduction of observers is not mentioned in literature. It is impossible to assume absolute motion and the assumption that all our observations are related to our positions, i.e. the position of the observer or Zuschauer, "dem Ort unseres Aufenthaltes Schätzen" [Euler E842, § 77]. Moreover, Euler analysed the observations of different observers who compare their theories and their measurements ("schätzen" is equivalent to "measure").

[47] This method had been later rediscovered and renewed by Einstein as a fundamental principle of physics.

[48] Euler's theory on the origin of forces is different both from Newton's and Leibniz's explanations of the origin of forces (compare Chap. 4).

[49] Descartes excluded the dualism between bodies and empty space (vacuum). Following Epicurus, bodies and empty space are related to each other since bodies need the empty space for executing their motion. However, instead of bodies moving in empty space, Descartes constructed the theory of vortices which was proofed as erroneous by Newton. Moreover, Newton concluded that there is

invented the basic idea of relative motion,[50] the dualism is not completely removed, but necessarily reappears if rest and motion are considered as properties of the bodies and not as relations between bodies. The ambiguity can only be removed by a consequent *relational* approach based on the distinction between uniform and non-uniform motion. Then, rest and uniform motion are related to the preservation of the state of bodies whereas non-uniform motion and the change from rest to motion are related to the change of the state [Euler E842, Anleitung, Kap. II–VI].[51]

> Addidi denique translationem illam fieri ex vicinia, non quorumlibet corporum contiguorum, sed eorum duntaxat, quod tanquam quiescentia spectantur. Ipsa enim translatio est reciproca, nec potest intelligi corpus AB transferri ex vicinia corporis CD, quin simul etiam intelligatur corpus CD transferri ex vicini corporis AB: ac planeeadem vis et actio requiritur ex una parte atque ex altera. [Descartes, Principia, Part II, XXVI]

Moreover, the velocity of a body becomes a quantity of arbitrary magnitude since its magnitude depends on the frame of reference. In modern physics rest and motion are not considered as a dualistic concept since all motions are described in terms of *relative* motion, however, other dualisms appeared. The famous example is the *wave – corpuscle* dualism known from quantum mechanics.[52] However, following

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not only an empty space, but a space without any relation to bodies and called this space the *absolute* space. Beside this absolute space Newton invented a relative space which is commonly used for practical purposes of measurement [Newton, Principia]. Following Newton, the absolute or mathematical space (time, motion) is related to thinking whereas the relative space (time, motion) is related to sensual experience.

[50] "25. What motion is, properly taken. But if we were to consider not so much from common usage as from the truth of the matter what should be understood by motion, in order that some determinate nature be assigned to it, we can say it is the translation of one part of matter, or of one body, from the vicinity of those bodies that are in direct contact with it and are viewed as at rest to the vicinity of others. Where by 'one body' or 'one part of matter' I understand everything that is transferred at the same time, even if this itself might again consist of many parts which have other motions in themselves. And I say that translation is not the force or action that transfers, as I shall show that this [motion] is always in the mobile, not in the mover, because these two are not usually distinguished with sufficient care, and that it is only its mode, not some thing subsisting [in it], just as shape is the mode of the shaped thing and rest [the mode] of the thing at rest." [Descartes, Principles (Mahoney), Book II, § 25]

[51] "Es ist unter den Schullehrern stark gestritten worden, ob die Bewegung unter die Eigenschaften (*proprietates*) oder Zufälligkeiten (*accidentiae*) eines Körpers gezählt werden müsse oder nicht? Eine Eigenschaft kann dieselbe nicht sein, weil die Eigenschaften eines Dinges unveränderlich sind; und da die Zufälligkeiten also erklärt werden, dass alle Veränderungen eines Dinges in den Zufälligkeiten vorgehen dergestalt, dass, wenn diese verändert werden, das Ding selbst eine Veränderung leide, so ist es klar, dass man die Bewegung auch nicht unter die Zufälligkeiten eines Körpers zählen könne. Man muss aber zweierlei Zufälligkeiten zugeben, davon die einen das Ding an und für sich selbst angehen, die anderen aber nur in seinem Verhältnisse mit anderen Dingen bestehen; und alsdann bleibt kein Zweifel übrig, dass nicht die Bewegung unter diese letztere Art von Zufälligkeiten zu rechnen sei. Solchergestalt fallen alle Schwierigkeiten weg, welche sowohl gegen die Bewegung selbst als die Mittheilung derselben vorgebracht zu werden pflegen." [Euler E842, § 18]

[52] It is necessary to stress that this dualism appeared even in 17th century between Newton's and Hyugens's theory of light and light propagation. Newton invented the theory of emanation of small

Heisenberg, the procedure to analyze the appearance of wave-particle dualisms follows the same rules which had been introduced before in case of *relative* motion. The behaviour of atomic systems should be analyzed in different experimental procedures, i.e. in its relations to other systems called apparatus. Then, it follows from the theory as Heisenberg claimed that the behaviour is to be necessarily different if the conditions for the measuring procedure are modified [Heisenberg, Anschaulich]. Moreover, the results are mutually connected by a general relation called uncertainty relation.

Following Descartes, the basic laws are those for the onservation of rest and motion, i.e. motion in a straight direction. The first law of motion is formulated without any reference to forces.

> XXXVII. Prima lex naturae: quod unaquaeque res, quantum in se est, semper in eodem statu perseveret; sicque quod semel movetur, semper moveri pergat. [Descartes, Principia, Book II, XXXVII][53]

Descartes' formulation of the theorem had been accepted by Newton. The vis insita, or innate force of matter, is a power of resisting, by which every body, "as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line". [Newton, Principia, Definitions]

> (...) ex hoc solo, quod Deus diversimode moverit partes materiae, cum primum illas creavit, jamque totam istam materiam conservet eodem e modo eademque ratione qua prius creavit, eum etiam tantundem motus in ipsa semper conservare a indivisa, manere, quantum in se est, in eodem semper statu, nec unquam mutari nisi a causis externis. Ita, si pars aliqua materiae sit quadrata, facile nobis persuademus illam perpetuo mansuram esse quadratam, nisi quid aliunde adveniat quod ejus figuram mutet. Si quiescat, non credimus illam unquam incepturam moveri, nisi ab aliqua causa ad id impellatur. [Descartes, Principia, Book II, XXXVI]

Descartes joined two models to explain the laws of motion, (i) the immutability of God and (ii) the properties of body "as much it lies in it". By the first theorem, the *conservation of total motion* is ensured whereas, by the second theorem, the redistribution of motion among the interacting bodies is properly described in terms of the inherent properties of bodies.[54] The reason for any change of the state is always of external origin. This Cartesian basic assumption forms the reliable foundation of mechanics. Descartes introduced the distinction between "internal" ("quantum in se est") and "external" ("manere, quantum in se est, in eodem semper statu, nec unquam mutari nisi a causis externis") [Descartes, Principles, Book II, XXXVI]

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particles from the surface of the sun whereas Huygens explained the propagation of light as the propagation of a wave (compare [Euler E343, Lettres XVII–XIX]).

[53] "XXXVII. The first law of nature is that each one thing, as much in it lies, may persist in same state, and thus which is moved, may go on always to be moved." [Descartes, Principles (Ross), Book II, XXXVII]

[54] Euler separated these two theorems by exclusion of any intervention by God or by ghosts: "there are no contributions to the changes in the world of bodies which are due to ghosts" [Euler E842, § 49] (compare Chap. 4). Obviously, this essential step had been prepared by Leibniz who claimed that there is no interaction or mutual or one-sided influence between souls and bodies [Leibniz, Monadology, § 80].

causes or principles. The distinction had been not only maintained, but also generalized, at first by Newton and Leibniz by splitting in a variety of correlated pairs of notions, mainly related to forces (compare Chap. 2) and later by Euler who made use of Descartes distinction for the analysis and definition of forces.[55] Following Descartes, it is the same "force" which ensures the action of bodies upon other bodies and guarantees the resistance again the action of other bodies.[56]

> Hic vero diligenter advertendum est, in quo consistat vis cujusque corporis ad agendum in aliud, vel ad actioni alterius resistendum: nempe in hoc uno, quod unaquaeque res tendat, quantum in se est, ad permanendum in eodem statu in quo est, juxta legem primo loco positam a. [Descartes, Principia, Book II, XLIII][57]

This principle had been partially preserved by Newton, but only completely recovered and applied to the investigation of the origin of forces by Euler. Newton and Leibniz completed the Cartesian system by the introduction of a variety of forces (compare Chap. 2). Although Leibniz had been rather overdone the introduction of various forces, he was, however, consequent and supposed that even the forces should be prior to extension [Leibniz, Specimen, I (1)].[58] In goal and spirit, Euler followed Leibniz and took advantage from the idea to reconsider Descartes' axiomatics and to replace extension with impenetrability [Euler E842, Capitel 5] (compare Chap. 4).[59] However, notwithstanding the considerable differences in the proposed theorems of the distinguished authors, all of them, the predecessors and the followers agreed in one cruicial point of reference which was unquestionable accepted by all of them, the decisive role of ancient science as a prototype for reliability, demonstration and rigor mainly represented by Euclid and Archimedes.

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[55] Euler distinguished between "internal" and "external" principles related to the inertia or the preservation of state and the generation of forces by bodies or the change of the state, respectively [Euler E015/016], [Euler E842], [Euler E289].

[56] "Corpus est extensum resistens. Extensum est quod habet magnitudinem et situm. Resistens est quod agit in id a quo patitur. Vacuum est extensum sine resistentia." [Leibniz, A VI, 267]

[57] "43. Hier ist genau zu beachten, worin die Kraft des Körpers bei seiner Wirksamkeit (agendum) auf einen anderen oder sein Widerstand gegen dessen Wirksamkeit (action) besteht; nämlich lediglich darin, dass jede Sache an sich so weit es an ihr ist (quantum in se est) strebt, in dem Zustand zu beharren, in dem sie ist, nach dem an erster Stelle aufgestellten Gesetze." [Descartes, Prinzipien, Buch II, 43]

[58] "In rebus corporeis esse aliquid praeter extensionem, imo extensione prius, alibi admonuimus, nempe ipsam vim naturae ubique ab Autore inditam." [Leibniz, Specimen, I (1)] "(...) quod in corpore praeter magnitudinem et impenetrabilitatem poni debeat aliquid, unde virium consideratio oriatur, cujus leges metaphysicas extensionis legibus addendo." [Leibniz, Specimen, I (11)]

[59] Châtelet, on the contrary, preserved the Cartesian frame and claimed that the extension is independent of forces. "147. Ces trois principes, sçavoir, l'étendue, la force passive, & la forces motrice, ne dépendent point l'un de l'autre; car ce sont les essentielles du Corps, & on a vu que les essentielles ne se déterminent point mutuellement, mais qu'elles peuvent seulement subsister ensemble sans se détruire. Ainsi, la force active & la force passive ne découlent point de l'étendue. (...) Il est aisé de voir que la force active ne résulte ni de l'étendue, ni de la force d'inertie (...). La force d'inertie ne peut être non plus la cause de la force active à laquelle elle résiste." [Châtelet, Institutions, § 147].

1.3 The Common Basis: The Ancient Prototype in Geometry and Mechanics

Archimedes' theory played a crucial role for the development of mechanics. Leibniz claimed that the ancients had the theory of "dead forces" whereas it is presently necessary to develop a theory of "living forces". Following Euler, the problem is properly formulated and completely solved if the relation between dead and living forces is analytically represented in terms of the calculus (compare Chap. 4). Euler referred to Maupertuis' principle for rest and motion [Maupertuis, Repos], [Maupertuis, Accord] and demonstrated the *Harmonie entre les principes generaux de repos et de mouvement de M. de Maupertuis* [Euler E197][60] or, as far as the disciplines are concerned, the harmony between static and mechanics represented by rest and motion.[61] Following Maupertuis, a system of interacting bodies remains in a stable configuration in the state of rest if the centre if gravity is positioned as low as possible [Maupertuis, Repos]. The ancient prototype for the modelling of general relations between bodies and forces is Archimedes' lever.[62] The decisive step for finishing the debate on the measure of living forces is the *analytic* representation of the subject in terms of the calculus [Euler, Mechanica], [d'Alembert, Traité].

Archimedes' lever is composed of two bodies connected rigidly to each other and the fulcrum (pivot). The fulcrum is fixed whereas the bodies can be moved. Due to their rigid connection the motion of one body B1 results always in a motion of the other body B2. Therefore, motion of B1 is always a motion *relatively* to B2

[60] This paper is part of a series of papers Euler devoted to the principle of least action and the investigation of forces [Euler E145], [Euler E146], [Euler E176], [Euler E181], [Euler E182], [Euler E186], [Euler E198], [Euler E199], [Euler E200], [Euler E343, Lettre LXXVIII].

[61] Lagrange (1736–1813) continued and developed Euler's program: "Je le divise en deux Parties: la Statique ou la Théorie de l'Équilibre, et la Dynamique ou la Théorie du Mouvement; et chacune de ces Parties traitera séparément des corps solides et de fluides. On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme." [Lagrange, Mécanique, Preface] published 1788. "On a conserve la notation ordinaire du Calcul différentiel, parce qu'elle répond au système des infiniment petits, adopte dans ce Traité. (...) ou par la méthode analytique des fonctions dérivées, on peut employer les infiniment petits comme un instrument pour abréger et à simplifier les démonstrations. C'est ainsi qu'on abrège les démonstrations des Anciens par la méthode des indivisibles." [Lagrange, Mécanique, Preface, Second Edition]

[62] "(...) ce qui ramène directement ce principe à celui du levier; (...)." [Lagrange, Mécanique, Preface, Second Edition] Jakob Bernoulli made use of the model of the lever and introduced a method nowadays called "virtual displacements" [J. Bernoulli, Acta Eruditorum (1691), p. 317]. "Yet another complication came with a 1691 *Acta Eruditorum* article by Jacob Bernoulli (1654–1705) on the compound pendulum's center of oscillation. Expressing discomfort with the 'obscure hypothesis' from which Huygens had derived his solution – though not denying it – Bernoulli offered an alternative derivation of the same result by replacing Huygens's hypothesis with the principle of the lever. That principle, unlike Huygens's, is one of static equilibrium, invoking only static forces and what we now call virtual displacements. Bernoulli used it to obtain an equilibrium condition along the pendulum's rigid string that yields the quantities of motion transferred from one bob to another." [Smith, http://www.physics.odu.edu/~kuhn/PHYS101/VisViva.html] Smith analyzed the prehistory, but did not comment on the development in the 18th century.

independently of the action from outside by which one of the bodies is set in motion. The same is true for the relative rest. An observer occupies (i) either a place which is different from the places occupied by the bodies and the fulcrum or (ii) replaces one of the bodies and represents a force which made the remaining body move.[63]

The equilibrium is formulated in terms of a two-body (weight) problem. Mechanically, both the bodies are *internal* and *equivalent* parts of the system. For any given weight A there is always a weight B to establish the equilibrium. The distance between the weight A and weight B is of the same magnitude as the distance between the weight B and the weight A, i.e. DIST(A, B) = DIST(B, A).[64] Before Archimedes people had been formulated the problem as follows:

> Why is it that small forces can move great weights by means of a lever, as was said at the beginning of the treatise, seeing that one naturally adds the weight of the lever? For surely the smaller weight is easier to move, and it is smaller without the lever. Is the lever the reason, being equivalent to a beam with a cord attached below, and divided into two equal parts? [https://www.math.nyu.edu/∼crorres/Archimedes/Lever/LeverIntro.html]

As far as motion is concerned the same question had been already adequately formulated by Leibniz, but inadequately answered by establishing a difference between the Archimedean principle of lever and the Cartesian principle of relative motion [Leibniz, Specimen, I (10)].[65]

> (…) atque adeo si necesse esset concursus eventum sola compositione conatuum Geometrica, ut explicuimus, determinari: tunc sequi debere, ut incurrentis, etiam minimi, conatus toti excipienti, licet maximo imprimatur; atque adeo maximum quiescens a quantulocunque incurrente sine ulla hujus retardatione abripiatur. [Leibniz, Specimen, I (10)]

The ancients had the science of rest or pressure or equilibrium or "dead forces" [Leibniz, Specimen, I (8)].[66] Therefore, Leibniz considered dynamics as the science of "living forces". Then, the requirement to establish a relation between "dead" and "living" forces is connected with the problem to *transfer* the principles valid for statics or rest to dynamics or motion (Fig. 1.1). The transfer of Archimedes law for the *distances* in case of equilibrium between two bodies (*weights*) to the case of relative motion of two bodies is established by replacing the *fixed* distances between the bodies by *variable* (changing) distances. Equilibrium is defined by the mutual compensation of all forces.

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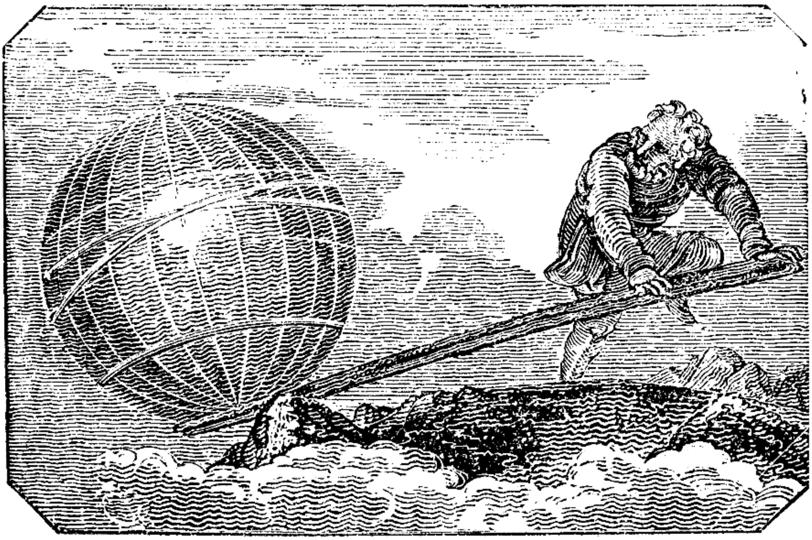
[63] Archimedes stated: "Give me a place to stand and I will move the earth." [Pappus] Therefore, the observer is an *actor* and, moreover, he becomes *a part of the system*. As it will be demonstrated, Euler also analyzed the role of the observer being a part of the modelled system which consists of bodies and observers.

[64] Descartes transferred this relation to the relative velocity.

[65] Châtelet accepted Leibniz's argumentation [Châtelet, Institutions, § 42].

[66] As Leibniz, also Newton referred to the ancient prototype in developing new methods in mechanics and mathematics. This affinity and correlation can be especially demonstrated for the development of the calculus by Newton. "And after this manner the Ancients by carrying moveable right Lines along immoveable ones in a Normal position or Situation (…)." [Newton, Quadrature (Harris)] (compare Chap. 3)

Transfer Principle



Body 1	Body 2	Body 3
Earth	Fulcrum	Archimedes
Earth	other Earth	Archimedes

$$m_1 \cdot l_1 = m_2 \cdot l_2$$

"Magnitudes are in equilibrium at distances reciprocally proportional to their weights." [Archimedes (Heath), Propositions 6 and 7 of Book I *On the Equilibrium of Planes*]

Fig. 1.1 Archimedes' principle [https://www.math.nyu.edu/∼crorres/Archimedes/Lever/Lever.jpg]

> Wir nähern uns dabei wieder mehr der Art der antiken Geometrie und lernen geradezu das Wesen der letzteren vermöge unserer modernen Auffassungen in neuer Weise zu verstehen. [Klein, Arithmetization][67]

Klein commented on a treatise by Zeuthen:

> Als Zweck der Schrift bezeichnet der Verfasser, 'eine geometrische Wiederherstellung des Inhaltes und des Zusammenhanges der antiken Lehre von den Kegelschnitten zu geben und zu begründen'.
>
> 'Der Darstellungsform der Alten fehlen die Eigenschaften, welche dieselbe zu einem bequemen Mittel hätte machen können,[68] den reichen Inhalt der griechischen Geometrie, geschweige denn die Arbeitsweise derselben, auf spätere Geschlechter zu übertragen.
>
> Trotz des Eifers, mit dem die Mathematiker der Renaissancezeit sich auf das Studium der Mathematik geworfen hatten, konnten sich dennoch während der ganzen neueren Zeit neue Einflüsse von Seiten der griechischen Geometrie noch geltend machen. (...) In unserer

[67] Quoted by Zeuthen [Zeuthen]. Zeuthen analyzed the role of geometric rigour [Zeuthen, Apollonius].

[68] Compare Euler's intention to make the methods applicable for the common reader by the development of an appropriate algorithm [Euler E015/016, Preface] (Compare Chap. 4).

> Zeit fahren dieselben Ursachen fort ihre Wirkung auszuüben, und selbst der, welcher sich einige Bekanntschaft mit den Schriften der Alten erworben hat, wird kaum geneigt sein, die in diesen gewonnenen Resultate genügend hoch zu würdigen, (...). Die Geometrie bei den Alten wurde nicht bloss ihrer selbst wegen entwickelt, sondern diente als Organ für die Grössenlehre'. [Zeuthen]

Geometry is closely related to static or the science of equilibrium and vice versa since the distances between the bodies are related to each other as commensurable quantities. The laws of equilibrium are the basis for the laws of motion. Euler made use of Archimedes' methodology [Euler E015/016, § 56]. The lever can be considered as a prototype of a *two-body* problem,[69] i.e. a model for a *relational* approach to mechanics or the science of motion which results in a relation between quantities represented in terms of *two* lengths and *two* masses (heavy masses or weights determined with respect to the "hidden" third mass of the earth), $m_1 l_1 = m_2 l_2$, or, between different bodies as we are told by Archimedes "if there is another earth".

> Archimedes, however, in writing to King Hiero, whose friend and near relation he was, had stated that given the force, any given weight might be moved, and even boasted, we are told, relying on the strength of demonstration, that if there were another earth, by going into it he could remove this. [Plutarch]

The relational approach in the case of motion is also based on a two mass (two body) model and results in a relation between the *two* masses m_1 and m_2 and the increments of velocity of *two* moving bodies or the changes in velocity, i.e. $m_1 dv_1 + m_2 dv_2 = 0$ (compare Chap. 4).[70] Obviously, the relation to the ancient prototype is detached if the motion of *one* body under the influence of an "impressed moving force" is studied. Then, the other body is "hidden" since the "force" is nothing else than the substitute for the other body. Following Archimedes who at first (A1) considered the lever in empty space and, at second transferred the result into real world (A2), Euler considered the *motion* of bodies at first in the empty space. In a preliminary step, Euler considered (i) one body in the empty space, (ii) two bodies interacting in empty space and (iii) transferred the result in real word [Euler E015/016, § 56] (compare Chap. 4 and Sect. 4.4). From the relation for the equilibrium its follows that the smallest mass can equilibrate the largest mass if the distance from the pivot or fulcrum is sufficiently increased. Euler transferred this model into mechanics or the science of motion.[71] Measuring the mass by the inertia the relation

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[69] However, most of the followers preferentially focused their attention at the lever, but much less at the two body problem behind. [https://www.math.nyu.edu/~crorres/Archimedes/Lever/LeverQuotes.html] In equilibrium, the bodies are resting relatively to each other and relatively to other bodies. In the absence of a rigid connection, the bodies used to move. However, in case of interaction by impact, the conditions for the lever model are reproduced since the bodies are also resting relatively to each other. Assuming Euler's model, the forces generated by bodies are described by the same relation as in the case of rest, $K_1 + K_2 = 0$.

[70] This relation is also used for the *operational* definition of mass which had been later proposed by Mach [Mach, Mechanik].

[71] The rejection of this Archimedean principle [Leibniz, Specimen, I (10)] may be considered as one the main shortcoming of Leibniz's theory of forces which is, however, a result of his concept of non-relational forces. Although Leibniz also assumed "relational" or "derivative forces", the *force*

between two masses is expressed in terms of the *change* in velocity.[72] The relation has to be formulated in terms of *two inert* masses (or, as in case of gravitation, in terms of two heavy masses) whose inertia has to be considered simultaneously.

1.3.1 Euclid, Archimedes, Heron

Searching for a foundation of the method of fluxions, Newton recovered the geometry related methodological basis by starting with the *generation of a line* by the *motion of a point* and the idea that time itself is also generated by a continual flux.

> I don't here consider Mathematical Quantities as composed of parts extremely small, but as generated by a continual motion. Lines are described, and by describing are generated, not by any apposition of parts, but by a continual motion of points (...), Time by a continual flux, and so in the rest. These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies. And after this manner the Ancients by carrying moveable right Lines along immovable ones in a Normal position or Situation, have taught us the Geneses of Rectangles [Newton, Quadrature, (Harris)]

In the treatise *Initia rerum mathematicarum metaphysica*, Leibniz also recovered the same methodological basis by starting with the generation of a line by the motion of a point stressing that the motion is *continuously* and *successively* performed.

> Motus est mutatio situs. (...) *Via* est locus continuus successivus rei mobilis. (...) *Superficies* est via Lineae. *Amplum* vel *Spatium* vel ut vulgo solidum est via superficiei. [Leibniz, Initia][73]

Following Tropfke, the basic ideas can be traced back to the comments of Heron and Proclus on Euclidean geometry which had been developed by the scholastic scholars.

> Über die Grundlagen der Geometrie ist noch oft im Altertum geschrieben worden; eingehenderen Bericht gibt uns darüber PKOKLOS (410–485 n. Chr.; Byzanz, Athen)[74] in seinem Kommentar zu dem ersten Buche der Elemente EUKLID's, Angelpunkte bleiben immer die Euklidischen und Heronischen Definitionen. Darüber hinaus kamen auch die arabischen Gelehrten kaum, wie uns der Kommentar des Arabers ANNAIEIZI (um 900 n. Chr.) zeigt. Die Scholastik des Mittelalters bevorzugte die Bewegungsdefinitionen und leitete aus ihnen

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as well as the *velocity* had been assigned as a *persistent quality* to one body instead of resulting from relative motion. As a consequence, the *relational* theory of time and space is not compatible with Leibniz's theory of forces. Following Leibniz, Châtelet used to split this setup into two different configurations, but considered only one of two relations: (i) body A as active and body B as passive and resisting and (ii) body B as active and A as passive [Châtelet, Institutions, § 142].

[72] Châtelet did not distinguish between velocity and change of velocity and argued as Leibniz did before in 1695 [Châtelet, Institutions, § 142], [Leibniz, Specimen, I (10)].

[73] The editor commented that the paper is the attempt to give a foundation of the calculus by the "mathesis universalis" which is a part of the "characteristica universalis" [Leibniz, Philosophische Schriften, Band 4, Edited by Herbert Herring, Suhrkamp]. Furthermore, the "characteristica universalis" is a part of Leibniz's logical foundation of his theory.

[74] See Proclus comments on Euclid [Proclus].

> den Begriff der Stetigkeit ab. Ganz modern klingt die Erklärung des JUBDANUS NEMOBARIUS (...), die aussagt, daß eine Linie ein stetiges Gebilde erster Ordnung, Fläche und Körper solche höherer Ordnung seien: *'Simplex autem continuitas in linea est, duplex quoque in superficie, triplex in corpore. Punctus est fixio simplicis continuitatis'*. [Tropfke, p. 39]

This development had been continued until the 17th century where Isaac Barrow made use of the theorem and Newton was educated by him. Although Newton later referred to Euclid,[75] the idea that the "line is a flux of a point" does not belong to Euclid. Hence, Newton went beyond the Euclidean frame by the introduction of fluents and fluxions, but re-established later Euclid's method arguing against Descartes.

> Proclus here, uses another definition which does not belong to Euclid 'the line is the flux of a point' and the line is the flux of a plane. Proclus tried to elucidate the simplicity of the first three postulates with the concept of movement. The demand of the postulates can be accorded the introduction of the movement. Nevertheless the introduction of the notion of movement is not so simple. Proclus tried to explain his choice. [Phili][76]

In developing the *Method of Fluxions*, Newton was promoted by Isaac Barrow.[77] The "long trip" of the ideas of flux and fluxion from ancient times to Barrow and finally to Newton had been reviewed by Hoppe [Hoppe][78] and Phili [Phili].[79] How-

[75] Pemberton commented on Newton's remark on Euclid's geometry: "(...) speaks with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclid." [Pemberton]

[76] Proclus: The Philosophical and Mathematical Commentaries of Proclus (transl. T. Taylor) 2 Vols. London. 1788 p. 123.

[77] "After a long trip through France, Italy, Smyrni, Constantinople, Venice, Germany and the Low Countries an English erudite became Regius Professor of Greek in the University of Cambridge. His name was Isaac Barrow and he was the person who culminated flux's concept and led his pupil Issac Newton to establish his theory of fluxions, his interpretation of the infinitesimal calculus, in an extended and systematic process. The Lectiones Geometricae of Barrow, although it reflected the philosophical nature of the writer, it dealt with the study of curves as generated by moving points and lines. The old derivation point – line – surface – solid, the ancient sequence has appeared again intact and fertile, and incorporated in the first official steps of the new born analysis. The Geometrical Lectures appeared in 1670 and J. M. Child deduced the opinion that were for the most part evolved during Barrow's professorship at Gresham College (July 1662–May 1664). (...) Thus, a line is considered to be the trace of a moving point. The germ of Geometrical Lectures found the appropriate ground. The student of Barrow, Isaac Newton, under the influence of his tutor, gave the final development in the flux's concept, formalizing and unifying the algorithmus of the calculus." [Phili]

[78] See also Klein [Klein, Elementarmathematik].

[79] "Many centuries later, circa 1290, Petri Philomeni de Dacia in Algorismum vulgarum Johannis de Sacrobosco Commentarius presents the sequence of generation point – line – surface by motion, so line generates a surface, a surface generates a solid. In the latin edition of Euclid's Elements by Christoph Clavius (1574) we found the word fluere for the description of the origin of lines and surfaces, by means of flowing points and lines, similar to Petrius. In the fourteenth century the study of mathematical sciences flourished in Oxford. Oresme used some concepts and terminology of Newton's theory of fluxions. The Merton calculators used the terms fluxus and fluens. But for Oresme these terms were related to geometrical representation. In Oresme's works exist two fundamental notions: firstly the representation of a physical quality by a surface; secondly the

ever, there are two approaches to represent the relations between points, lines, surfaces and solids. The traditional order beginning with the point and ending with the solid (space, amplum) was preserved by Newton in 1687 [Newton, Principia] and also by Leibniz in 1715 [Leibniz, Initia]. The alternative approach by starting with the space (solid, amplum) had been already discussed in ancient time by Aristotle and renewed in the middle ages. Following Aristotle [Aristotle, Metaphysics], solid, plane and line are distinguished from each other by the different modes of division being threefold, twofold and simple, respectively. Hence, the different geometrical objects are *operationally* distinguished from each other by their capacity of subsequent divisions. However, the procedure cannot be terminated by obtaining an indivisible object or a point [Euler E842, Chap.2].[80]

Aristotle stated:

> (...) Thus that which is quantitatively and quantitative wholly indivisible and has no position is called a unit; and that which is wholly indivisible and has position, a point; that which is divisible in one sense, a line; in two senses, a plane; and that which is quantitatively divisible in all three senses, a body. And reversely that which is divisible in two senses is a plane, and in one sense a line; and that which is in no sense quantitatively divisible is a point or a unit; if it has no position, a unit, and if it has position, a point. [Aristotle, (Ross) Metaphysics, 1016 b 20–27]

Nevertheless as Leibniz claimed and Newton also assumed, the line or the continuum does not consist of points [Leibniz, Theodizee], but is generated by the motion of a point [Newton, Quadrature], [Leibniz, Initia]. Euclid assumed indirectly the existence of objects of lower as well as of higher dimension, i.e. points having no length and planes having length and breadth.

> A line is a length without breadth. [Euclid, Elements, I, Def. 2]

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concept of a surface as the flux or motion parallel to itself. Oresme gave emphasis to the description of lines, surfaces and solid by motion. *Napier* in 1614, in his Descriptio, employed the idea of the fluxion of a quantity to represent by means of lines the relations between logarithms and numbers. ‘Sit punctus A a quo decenda sit linea fluxu alterius puncti, qui sit B; fluat ergo primo momento B ab B in C, secundo momento C in D etc.’ Cavalieri followed this trend in holding that surfaces and volumes could be regarded as generated by the flowing indivisibles. ‘Communes sectiones talis moti sive fluntis plani et figurae”, “planum motu seu fluxus’. The flowing motion in Cavalieri plays a relatively minor role as he did not develop this idea into geometrical method. This was done by his successor Torricelli. Toricelli considered the curves generated by a point which moves along a uniformly rotating line with a velocity, not necessarily uniform. Roberval also regarded every curve as the path of a moving point and accepted as an axiom that the direction of motion is also that of the tangent. Although Torriceli’s results remained unpublished his pupils and associates Angeli (1623–1697) and Ricci (1619–1682) were able to continue his research.” [Phili]

[80] “In bewußter Absicht, anschaulicher zu sein, ändert *Gerbert* (940 Auverge – 1003 Rom, seit 999 Papst *Sylvester* II.) die Reihenfolge der Euklidischen Definitionen und beginnt mit dem Körper. (Prudentibus (...) taediosum non sit, si a solido corpore, quod communi hominum sensui notios est, praepostero incipiens ordinesimplicioribus, quid haec singula sint, paucis tentabo monstrare). In neuester Zeit ist diese Umstellung bekanntlich ein Unterrichtsprinzip geworden. *Gerbert* hebt ausdrücklich hervor, erstmalig in der Literatur, daß Punkte, Linien, Flächen ohne Körper nicht existieren und nur mit dem Verstande als unkörperlich, gleichsam außerhalb der Körper bestehend, erfaßt werden können.” [Tropfke, p. 41]

Therefore, the line is neither a point nor a plane. It is not necessary to refer to the space. The suggestive supremacy of intuition was in power not only until Newton's time, but also later in the 19th century.[81] It had already been emphasized by Proclus.[82]

A completely new approach had been invented by Galileo who operationally or experimentally resolved the continuous motion into a series of discrete portions or parts of a straight line which are related to an as discrete series of time intervals. As a consequence, continuous motion had been projected onto the series of natural numbers and was described by the differences between consecutive terms of these series.

1.3.2 Galileo: A New Science Dealing with an Ancient Subject

Galileo joined the theoretical construction of motion and the experimental observation. Following the ancient classification of motion into natural and forced motion, Galileo demonstrated for the first time that the combination of both types of motion in case of a projectile is geometrically described by a parabola [Galileo, Discorsi, Third Day]. Although Galileo stepped beyond the ancient prototype in mechanics he preserved essential elements of the geometrical representation of motion. Ideal motions are performed along circles, i.e. closed lines of finite size.[83] The Galilean law of the continuation of motion in the absence of forces takes only into account the magnitude of velocity, but not the straight direction of motion where motion is continued in infinity. This axiom had been invented by Descartes [Descartes, Principles].

Galileo distinguished between *natural* and *forced* motion, i.e. between those motions like the motion of a falling body which are performed without the intervention from outside and those which are generated due to the intervention from outside by men like in case of motion of projectiles, respectively.

> My purpose is to set forth a very new science dealing with a very ancient subject. There is, in nature, perhaps nothing older than motion, concerning which the books written by

[81] "Allmählich erst erwacht wieder eine strengere Kritik, welche nach der logischen Berechtigung der kühnen Entwicklungen fragt, (...). Dies ist die Periode von *Gauss* und *Abel*, von Cauchy und *Dirichlet*. Aber hierbei ist es nicht geblieben. Bei *Gauss* wird die Raumanschauung, insbesondere die Stetigkeit des Raumes noch unbedenklich als Beweisgrund benutzt. (...) Daher die Forderung *ausschließlich arithmetischer Beweisführung*." [Klein, Arithmetization]

[82] " 'This definition,' says Proclus (pp. 97, 8–13), 'is a perfect one as showing the essence of the line: he who called it the flux of a point seems to define it from its genetic cause, and it is not every line that he sets before us, but only the immaterial line; for it is this that is produced by the point, which, though itself indivisible, is the cause of the existence of things divisible'." [Euclid (Heath) Preface]

[83] In the 20th century, these typical features in the change of paradigms or prototypes where new elements appeared in combination with essential elements of the former paradigm can be observed in the development of quantum mechanics (compare Chap. 8).

> philosophers are neither few nor small; nevertheless I have discovered by experiment some properties of it which are worth knowing and which have not hitherto been either observed or demonstrated. Some superficial observations have been made, as, for instance, that the natural[84] motion [naturalem motum] of a heavy falling body is continuously accelerated; but to just what extent this acceleration occurs has not yet been announced; for so far as I know, no one has yet pointed out that the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity. It has been observed that missiles and projectiles describe a curved path of some sort; however no one has pointed out the fact that this path is a parabola. But this and other facts, not few in number or less worth knowing, I have succeeded in proving; and what I consider more important, there have been opened up to this vast and most excellent science, of which my work is merely the beginning, ways and means by which other minds more acute than mine will explore its remote corners.
>
> This discussion is divided into three parts; the first part deals with motion which is steady or uniform; the second treats of motion as we find it accelerated in nature; the third deals with the so-called violent motions and with projectiles. [Galileo, Discorsi, Third Day][85]

Galileo represented mechanical laws in terms (i) geometry by a parabola and (ii) by an arithmetical progressions in terms of natural numbers. The latter representation is independent of geometry, but is solely based on measurements of finite *temporal* and *spatial* intervals. The correlation is mechanically due to the motion of bodies and, arithmetically due to the relation between the series of terms and the differences between the terms of these series. Time is represented by an arithmetical series of finite intervals whose differences are equal among each other. Hence, the *arithmetical* result may be *mechanically* interpreted by the theorem that "time is equable flowing". "Time" does not consist of these "equal parts" since the lengths of the time intervals can be arbitrarily chosen, but the *relation* between the lengths of consecutive intervals depends of the experimental setup. In case of a uniformly moving body or a substitute for such a body, this equality cannot be arbitrarily established or modified, but is due to the properties and the motion of the body [Euler E149] (compare Chap. 4) (Table 1.1).

[84] The distinction between natural and forced motion had been later removed by Euler who assumed that the subject of mechanics are the motions and the changes in motion which *are only caused by the bodies themselves* without interventions by ghosts [Euler E842, § 49] (compare Chap. 4). "49. *Alle Veränderungen, welche in der Welt an den Körpern vorgehen, insofern dazu von Geistern nichts beigetragen wird, werden von den Kräften der Undurchdringlichkeit der Körper hervorgebracht, und finden also in den Körpern keine anderen als diese Kräfte statt*." [Euler E842, § 49] Therefore, the name of "natural motion" is more appropriate than the name of "free motion" which had been chosen by the translators. "Se ne rilevano alcune più immediate, come quella, ad esempio, che il moto naturale dei gravi discendenti accelera continuamente; (...)." [Galileo, Discorsi, Third Day]

[85] In 1687, Newton indirectly referred to Galileo on explaining the motion of projectiles. "*Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon*. PROJECTILES persevere in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in more free spaces, preserve their motions both progressive and circular for a much longer time." [Newton, Principia, Axioms] (compare Chap. 2)

Table 1.1 Galileo's representation of motion in terms of series of integers

Time:	0		1		2		3		4		…	n
Path:	0		1		4		9		16		…	n·n
Diff:		1		3		5		7			…	2n − 1

Table 1.2 Galileo's representation of motion in terms of differences

Time:	0		1		2		3		4		…	n	t
Diff 1st:		1		1		1		1			…	1	Δt
Diff 2nd:			0		0		0		0		…	0	$\Delta\Delta t$

The differences represent the increments of the path in consecutive time intervals. Time and the continuous change of time (increase of time) are represented by intervals of the *equal* length, $\Delta t,\ 2\Delta t,\ 3\Delta t,\ldots$ where the first differences between two terms of the series are constant Δt (Table 1.2).

Hence, the intervals of the length Δt are separated neither by any intervals of finite length nor, following Leibniz (compare Chap. 3), by any intervals of infinitesimal length.[86] Although the increase is modelled by the discrete series of integers, time is not regarded to be a discrete variable since the *choice of the length* of the elementary interval or the unit Δt is neither a system related[87] nor a universal quantity. The continuous flow of time is assumed to be indepedent of the falling body [Newton, Principia, Definitions]. Consecutive time intervals of arbitrary, but equal lengths are not seperated by intervals of any finite length. In case of periodic motion, the system related parameters are given by

$$\nu_{\text{pend}} = \sqrt{g/l}, \qquad \nu_{\text{osc}} = \sqrt{k/m} \tag{1.1}$$

where, in case of the pendulum, g and l are the constant acceleration in the gravitation field and the length of the pendulum, respectively, whereas, in case of the harmonic oscillator, k and m are the force constant and the mass, respectively.

Following Galileo, the border (terminus) between consecutive temporal or spatial intervals is not defined by a continuous change of the finite temporal intervals or a continuous diminution of their length, being finally down scaled to evanescence (and the result is described in terms of the evanescent quantities), but *arithmetically* defined the absence of any *assignable* difference between consecutive intervals $\Delta t_i - \Delta t_{i\pm1} = 0$. Using the notion of function in Euler's interpretation [Euler E044], [Euler E015/016], time had been introduced as *independent variable* by Galileo. Then, the requirement of the *invariance* of the time intervals had to be fulfilled $\Delta\Delta t = 0$. This requirement concerns the theoretical model. It is quite different from the experimental realization. However, the experiments can only be

[86] Leibniz's foundation of the calculus will be considered in Chap. 3. Leibniz assumed that the differential *da* of a finite quantity is equal to zero.

[87] A *system related finite interval* of time appears only for periodic motions, e.g. in the case of pendulum where the same configuration of the system is reproduced after a certain time. In the mechanical model, the difference between consecutive intervals is also equal to zero. Hence, there is nothing "between" consecutive intervals except the "common border".

interpreted in the frame of the theoretical model if the same conditions are established for the experimental setup.

The same analytical procedure had been applied to the distances the falling is travelling. For uniform motions, the increment of the path $\Delta s \sim \Delta t$ is independent of the length of the time interval, i.e. for "any equal time intervals", proportional to the time interval, thus, by "steady or uniform motion, I mean one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal"[88] or $\Delta s = v \cdot \Delta t$ with $v = \text{const}$. Analytically represented, the difference between consecutive path interval is zero $\Delta x_i - \Delta x_{i\pm 1} = 0$ or $\Delta\Delta x = 0$.

Developing the *calculus of increments* and the *calculus of differences*, Taylor [Taylor, Methodus] and Euler [Euler E212] constructed the proper mathematical theory which fits perfectly the needs of the Galilean procedure to project the continuous motion onto the set of natural numbers. Continuity remains to be the basic item and the Leibnizian principle of continuity[89] remains to be in power, but it had been generalized to be also valid for discrete series of quantities [Euler E212, § 85]. Hence, Euler paved the way for a development which had been later called the *arithmetization of mathematics* [Klein, Arithmetization].

1.4 The New Prototype: Arithmetization of Mathematics and Mechanics

In 1893, Klein reviewed the development of the foundation of mathematics, especially the foundation of the calculus from Newton and Leibniz to Weierstraß. Klein acknowledged the pioneering work of the pathfinders Newton and Leibniz who developed mathematics in correlation to the "observation of nature". Not accidentally, the main unifying overriding principle had been assumed to be the *principle of continuity* by Leibniz who transferred it from geometry to physics[90] [Leibniz, Specimen, II (4)]. First of all, Klein highly acknowledged Euler's concept of functions and the calculus of differences introduced by Taylor and Euler.

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88 "UNIFORM MOTION. In dealing with steady or uniform motion, we need a single definition which I give as follows: DEFINITION. By steady or uniform motion, I mean one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal. CAUTION. We must add to the old definition (which defined steady motion simply as one in which equal distances are traversed in equal times) the word "any," meaning by this, all equal intervals of time; for it may happen that the moving body will traverse equal distances during some equal intervals of time and yet the distances traversed during some small portion of these time-intervals may not be equal, even though the time intervals be equal." [Galileo, Discorsi, Third Day]

89 "Indes ist der Geist, aus dem die moderne Mathematik geboren wurde, ein ganz anderer. Von der Naturbeobachtung ausgehend, hat er ein philosophisches Prinzip, das Prinzip der Stetigkeit, an die Spitze gestellt. So ist es bei den großen Bahnbrechern bei Newton und Leibniz, so ist es das ganze 18. Jahrhundert hindurch, welches für die Entwicklung der Mathematik recht eigentlich ein Jahrhundert der Entdeckungen gewesen ist." [Klein, Arithmetization]

90 By this procedure, Leibniz intentionally made use of *geometrical* rigour for physics, notwithstanding, that additional principles are needed which had been called "metaphysical" or "contingent" principles [Leibniz, Specimen], [Leibniz, Discours].

> Wir wollen nur, daß der allgemeine Funktionsbegriff in der einen oder anderen Eulerschen Auffassung den ganzen mathematischen Unterricht der höheren Schulen wie ein Ferment durchdringe. [Klein, Elementarmathematik, p. 221]

Following Klein, in the foundation and application of the *Method of Increments*, Taylor [Taylor, Methodus] made use of those infinitesimal quantities whose existence had been categorically excluded by Newton [Klein, Elementarmathematik].[91] Klein acknowledged the tight connection between the calculus of differences and the differential calculus which had been presented by Taylor and Euler.

> Ich möchte zuerst darauf hinweisen, daß das von Taylor zwischen Differenzen- und Differentialrechnung geknüpfte Band noch lange Zeit gehalten hat: Noch in den analytischen Entwicklungen Eulers gehen beide Disziplinen Hand in Hand, und die Formalen der Differentialrechnung erscheinen als Grenzfälle ganz elementarer Beziehungen, die in der Differenzenrechnung statthaben. Diese so naturgemäße Verbindung wurde erst durch die wiederholt erwähnten formalen Definitionen des Lagrangeschen *Derivationskalküls* aufgehoben. (...) In diesen Gedankenkreisen konnte Lacroix natürlich die Differenzenrechnung als Ausgangspunkt nicht mehr benutzen; sie erscheint ihm aber doch für die Praxis zu wichtig, als daß er sie weglassen wollte, und so ergreift er denn den Ausweg, sie in ganz selbständiger, übrigens ausführlicher Darstellung hinterher im dritten Bande zu bringen, ohne daß gedankliche Brücken von ihr zur Differentialrechnung führen. [Klein, Elementarmathematik, p. 253]

Klein emphasized that there was not only a "natural connection", but also a mutual correlation between the "formalism of these disciplines" which had been cut by Lagrange. Obviously with regret at this unpleasant disconnection, Klein commented that the further development resulted in a complete separation of these two directions and, despite the promising beginnings, the previous connection had been finally completely divorced. However, many contemporaries of Klein were pleased of disappearance of differentials in the end of the 19th century (compare Chap. 5).[92]

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[91] "Hierin liegt nun tatsächlich ein *Grenzübergang von unerhörter Kühnheit*. Hier operiert also Taylor im Grunde mit unendlich kleinen Größen (Differentialen) ebenso unbedenklich wie es die Leibnizianer taten; es ist interessant, sich zu vergegenwärtigen, daß er als ganz junger Mann von 29 Jahren noch unter den Augen Newtons von dessen Grenzmethode abwich. Freilich gelang ihm dadurch auch diese Entdeckung allerersten Ranges. Diese Bezeichnungen sind genau denen der Differentialrechnung analog, nur daß es sich hier um bestimmte endliche Größen handelt und von Grenzprozessen nicht die Redeist." [Klein, Elementarmathematik. pp. 250–251]

[92] The development is mirrored in the edition of *Meyers Lexikon* from 1895 and 1913 [Meyer].

Chapter 2
Newton and Leibniz on Time, Space and Forces

Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

Newton, Principia; Definitions

Si plures ponantur existere rerum status, nihil oppositum involventes, dicentur existere simul. Et ideo quicquid existit alteri existenti aut simul est aut prius aut posterius. Si eorum quae non sunt simul unum rationem alterius involvat, illud prius, hoc posterius habetur.

Leibniz, Initia

Further: Space and Time are Quantities: which Situation and Order are not.

Leibniz Clarke (Alexander), Clarke, 3rd Letter to Leibniz

The contemporary 20th century physics is based on a foundation which had been mainly invented by two scholars living 300 years ago in the 17th century, Newton and Leibniz, who assembled new rules for mathematics and mechanics being disparat from those known before them. Newton and Leibniz surpassed their famous predecessors and contemporaries Galileo, Huygens and Descartes. Both scholars composed an impressive and long-lasting system of concepts which survived all changes and modifications introduced later by their followers. It took over two centuries until a change comparable in significance and explanatory power had been invented in the beginning of the 20th century by Planck, Einstein, Bohr, Heisenberg and Schrödinger. Although only their contributions to mathematics and physics formed a reliable basis for the development in the 18th and 19th centuries, their theories of time, space and motion were thought to be essentially different from each other from the very beginning until present time.[1] Hence, the picture generated in

[1] Compare the summaries by Clarke [Leibniz Clarke], Mach [Mach, Mechanik], Reichenbach [Reichenbach, Space and Time].

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the 18th century was transmitted through the centuries. It is dominated by the opposition and difference and only step by step it had been discovered that Newton's and Leibniz's methods are grounded on a hidden relationship of common methods of thinking which had been only differently represented. This hidden conjunction and entanglement is clearly demonstrated by the controversy about the invention of the calculus [Newton, Principia, Book II, Sect. II, Lemma, pp. 250–254],[2] [Newton (Collins), Commercium].

Nowadays, the result of retrospection is summarized in terms of *equivalence* and *priority*, these words Meli had chosen for entitling his treatise on the Newton-Leibniz debate concerning the invention of the calculus [Meli]. Obviously, the well-known battles on the priority could only be set into scene because there was an equivalent approach and nobody else except Newton and Leibniz created equivalent alternative theories or foundations. A new approach had been only invented in the 18th century by Taylor, MacLaurin, Euler, d'Alembert and Lagrange (compare Chaps. 3 and 5).[3]

Newton and Leibniz almost simultaneously created the calculus between 1665 and 1675 with a negligible delay of the years compared to the delay in publication of the results published 20 and even 40 years later entitled *Nova methodus* [Leibniz, Nova Methodus] in 1684 and the preliminary announcement of the *Method of Fluxions* [Newton, Opticks] in 1704, respectively. Even in 1684, the impact of the new method was negligible and only understood by distinguished scholars Jakob and Johann Bernoulli. However, also those distinguished scholars like Jakob and Johann Bernoulli and later L'Hospital, Varignon and other mathematicians were asking for explanation by Leibniz [Bernoulli, Letter to Leibniz]. Only, as it could be expected, Newton had no problems of understanding, but supported Leibniz's approach by his authority. In 1687, Newton commented on the new method in *Principia* [Newton, Principia, Book II, Sect. II, pp. 250–253] although most of the results were geometrically demonstrated.

The same simultaneity in creation of new approaches appeared even more pronounced in making public the new foundation of *mechanics* by Leibniz [Leibniz, Brevis] and Newton [Newton, Principia] in 1686 and 1687, respectively. The common origin of the short communication by Leibniz entitled *Brevis demonstratio erroris memorabilis Cartesii et aliorum* [Leibniz, Brevis] and the comprehensive treatise by Newton entitled *Philosophiae naturalis principia mathematica* [Newton, Principia] is the criticism of Descartes's principles (compare Chap. 1). The main result of this exceptional breakthrough is that Newton and Leibniz did not only invent new programs for mechanics and mathematics, but completed also essential parts of their programs demonstrating the power of the new methods in solving problems which had been unavailingly attacked before. Even at

[2] "In literis quae mihi cum Geometra peritissimo G. G. Leibnitio annis abhinc decem intercedebant (...) rescripsit Vir Clarissimus se quoq; in ejusmodi methodum incidisse, & methodum suam communicavit a mea vix abludentem praeter quam in verborum & notarum formulis." [Newton, Principia, Book II, Sect. II, Lemma II, Scholion, pp. 253–254] (compare Chap. 3) Newton confirmed the equivalence of the approaches without emphasizing the differences.

[3] For the criticism by Nieuwentijt and Berkeley compare Chaps. 3 and 5.

present, in the 21st century, Newton's axioms of mechanics, Leibniz's living forces (presently known as kinetic energy) and the representation of calculus are still in power forming the indispensable basis of contemporary science. Also classical mechanics survived the advent of quantum mechanics [Planck 1900], [Einstein, Heuristisch], [Bohr 1 to 4] and becomes an essential part of Bohr's correspondence principle [Bohr, Correspondence] and had been accepted as a basic part of physics.[4]

The same story had to be told for Newton's and Leibniz's calculus. Leibniz's formalism and algorithmic representation of the calculus survived all attacks over centuries which had been initiated by Nieuwentijt [Nieuwentijt, Analysis] in 1695 and in 1696 [Nieuwentijt, Considerationes] and renewed by Berkeley [Berkeley, Analyst] in 1734. Even Leibniz's own interpretation of differentials as fictitious quantities [Leibniz (1712)] did not reduce the importance and reliability of the method. Finally, the foundation in terms of limits initiated by d'Alembert [d'Alembert, Encyclopédie, Limit], completed by Cauchy [Cauchy] and Weierstraß [Weierstraß] or Lagrange's alternative algebraic approach [Lagrange, Fonctions] confirmed the *validity* of the *algorithms* without changing the basic rules of the generation of derivatives and integration of functions (compare Chaps. 3 and 5). Unavoidably and mainly caused by the incomplete and retarded publication of essential papers, the reception of the legacy of Newton and Leibniz was unintentionally selective and unfinished over a long period.

Nowadays, in view of the progress in science and the diversity of problems appearing due to the variety of approaches and proposed models especially in the basic research in physics, people intended to complete the reception by analyzing the other parts of the legacy [Keynes], [Keynes (Reagle)], [Truesdell], [Dijksterhuis], [Westfall, Never], [Wilczek 2004a],[5] [Smolin][6] or even reconsidering of the well-known parts [Chandrasekhar] which may be regarded as a renewing of the development which also took place in the beginning of the 18th century.

Leibniz stated that the "ancients had the science of equilibrium", but it is necessary to have a "science of motion" [Leibniz, Specimen, I (8)] where "motion" is related to the "phenomena". The world is full of phenomena experienced by men who

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[4] Classical mechanics is necessary for the formulation of quantum mechanics (compare Chap. 8). "It is in principle impossible, however, to formulate the basic concepts of quantum mechanics without using classical mechanics." [Landau/Lifschitz, Quantum]

[5] "When I was a student, the subject that gave me the most trouble was classical mechanics. That always struck me as peculiar, because I had no trouble learning more advanced subjects, which were supposed to be harder. (...) Coming from mathematics, I was expecting an algorithm. (...) To anyone who reflects on it, it soon becomes clear that $F = ma$ by itself does not provide an algorithm for constructing the mechanics of the world." [Wilczek 2004a] As is will be demonstrated, the missing algorithm had already been established by Euler (compare Chap. 4).

[6] "This essay is written with the hope that perhaps some who have avoided thinking about background independent theories might consider doing so now. To aid those who might be so inclined, in the next section I give a sketch of how the absolute/relational debate has shaped the history of physics since before the time of Newton. Then, in Section 3 I explain precisely what is meant by relational and absolute theories. Section 4 asks whether general relativity is a relational theory and explains why the answer is: partly. We then describe, in Section 5, several relational approaches to quantum gravity" [Smolin].

intended to discover the principles behind the phenomena, i.e. the true principles the construction of the world is based on which are ruling the phenomena. The common name of these principles was chosen to be “forces of nature”. In the following decades in the 17th century and subsequently over three centuries until today there is a long-lasting debate about the *nature*, the *origin* and the *effects* of these forces which causes motions and the change of motion of the bodies. The debate can be traced back to the ancient science. Aristotle invented a correlation between motion described in terms of velocity and forces. Archimedes considered the equilibrium of forces for the model of the lever. Galileo invented an alternative model and discovered that uniform motion takes place without a moving force behind the body causing a change of position [Galileo, Discorsi].

Following Newton, the phenomena and the forces are related to each other [Newton, Principa]. Following Leibniz, the phenomena are described in terms of relative motion whereas the forces have to be related to substances [Leibniz, Specimen]. Leibniz developed a relational theory of time and space and a non-relational theory of forces.[7] However, Leibniz assumed a close correlation between motion (velocity) and forces. As a consequence, the “living forces” of bodies are assigned to moving bodies independently of the type of motion. Only Euler based mechanics on a force-free uniform motion and excluded the “force of inertia” [Euler E842] (compare Sect. 4). In the 19th century, the triumvirate of Descartes, Newton and Leibniz, who represented different scientific disciplines including theology in personal union, had been separated into three parts of philosophy (called metaphysics including theology), mathematics and physics. People demonstrated the errors of Descartes which had been already analyzed by Newton without acknowledging his merits in mathematics and mechanics. Moreover, they did not mention the life-long battle of Newton against the Cartesian methods in philosophy and mathematics (compare Westfall [Westfall, Never]). In the second half of the 19th century, this commonly accepted picture established by Mach and other authors was questioned by Russell [Russell, Western] and Couturat [Couturat, Leibniz] who discovered Leibniz’s contribution to logics and by Helmholtz [Helmholtz, Vorlesungen] who acknowledged Leibniz’s contributions to mechanics as far as the conservation of living forces and, generally, the energy is concerned. In view of the Leibniz’s manuscripts published by Couturat, Russell distinguished between the “exoteric” and the “esoteric” Leibniz to demonstrate that Leibniz was not only the author of the idea of the “best of all possible worlds”, but a thinker who based his metaphysics on logics [Russell, Western]. Therefore, Voltaire in the 18th century and Mach in the 19th century had been misled themselves because they did not recognize the complete Leibnizian system foreshadowed behind its

[7] Only Euler based mechanics on a *relational theory of motion and* a *relational theory of forces* [Euler E177], [Euler E181], [Euler E842] (compare Chap. 4). Obviously, although Euler assumed *relative motion*, the theory of time and space is mainly that of Newton’s absolute time and space. In the non-relativistic version of mechanics, r*elative* motion is always possible if the frame of reference is formed by two bodies which are localized in an immobile space or moving uniformly in one direction. Then, the transformation is the Galileo transformation. This kind of motion had been analyzed by Euler (compare Sect. 6).

official representation. In the 20th century, an analysis following in goal and spirit Russell's attempts had been given by Keynes for Newton's writings and activities [Keynes].[8]

Upon this background which was extended in the last decades by studying of Newton's complete writings,[9] the traditionally presupposed opposition and antagonism between the antipodes Newton and Leibniz optimally represented in the letters exchanged in the famous Leibniz-Clarke debate [Leibniz Clarke],[10] may be partially or even completely lifted taking into account the "esoteric components" in the work of Newton and Leibniz.

Newton's basic papers on motion and forces of bodies are: *De motu corporum* [Newton, De motu] written in 1685, *Philosophiae naturalis principia mathematica* [Newton, Principia] published 1687, the invention of the method of fluxions 1665–1671 [Newton, Method of Fluxions], letters to Leibniz 1665 and 1669

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[8] "Magic was quite forgotten. He has become the Sage and Monarch of the Age of Reason. The Sir Isaac Newton of orthodox tradition – the eighteenth-century Sir Isaac, so remote from the child magician born in the first half of the seventeenth century – was being built up. Voltaire returning from his trip to London was able to report of Sir Isaac – 'it was his peculiar felicity, not only to be born in a country of liberty, but in an Age when all scholastic impertinences were banished from the World. Reason alone was cultivated and Mankind could only be his Pupil, not his Enemy.' Newton, whose secret heresies and scholastic superstitions it had been the study of a lifetime to conceal!" [Keynes] [http://www-history.mcs.st-and.ac.uk/Extras/Keynes_Newton.html] The text was written in 1936.

[9] See The Newton Project [http://www.newtonproject.sussex.ac.uk/links.html] and [http://www.isaac-newton.org] and the links therein.

[10] Assuming Keynes' terminology, the published text may be considered as a debate between the "exoteric Newton" represented by Clarke and the "exoteric Leibniz". However, following Keynes, the debate was running on a *hidden subtext* both the opponents were acquainted with. A similar procedure was already observed for the invention of the calculus. The technical procedures and algorithms were of the same algebraic structure, but the foundations of the calculus were quite different and even in direct opposition to each other as far as the distinction between "sums and differences" (Cavalieri, Leibniz) and "fluents, moments and fluxions" are concerned (compare Chap. 3). In the debate on the priority in invention of the mathematical method, both authors were forced to disclose much more of their intentions as they primarily intended to do (compare Chap. 3). It should be stressed that both authors published the method only with a time delay of *several decades*, Leibniz in 1684 and Newton not until 1704. The delay may be explained by the common practice of publication, but may be also explained by the severe consequences for the interpretation of mechanics as the basic science for the understanding of the construction of the bodies and the ruling principles of the construction of universe. The intrinsic capability of *destructive* as well as *constructive* power of the Newton-Leibniz calculus was readily detected by critics who shortly after appeared [Nieuwentijt, Analysis], [Berkeley, Analyst] (compare Sect. 2.3). In 1734 for the first time and without any restraints, Euler stressed the *constructive* power [Euler E015/016], [Euler E212] whereas, in the same year in the treatise *The Analyst a Discourse addressed to an Infidel Mathematician*, Berkeley attacked the *destructive* potential [Berkeley, Analyst]. The debate had been continued in the 19th century where Cantor [Cantor] referred to the "exoteric Leibnizian interpretation" and in the 20th century where Robinson [Keisler], like Euler in the 18th century [Euler E212, § 85], recovered the power of Leibniz's principle of continuity as a ruling methodological principle (see Chaps. 3 and 5).

[Newton, Letter to Leibniz]. Newton published extended papers on the calculus only twenty year after Leibniz's *Nova Methodus* in 1704.[11]

Leibniz's basic papers on the motion and forces of bodies are: *Hypothesis physica nova* 1671 [Leibniz, Physica nova], comprising *Theoria motus abstracti* and *Theoria motus concreti*, the early manuscripts on the calculus [Gerhardt, Leibniz], [Gerhardt, Historia], [Child], *Tentamen anagogicum* [Leibniz, Tentamen], *Pacidius Philalethi* [Leibniz, Pacidius], *Nova Methodus* 1684 [Leibniz, Nova Methodus], *Brevis demonstratio* 1686 [Leibniz, Brevis], *Phoranomus* 1689 [Leibniz, Phoranomus], *Specimen dynamicum* 1695 and 1695 [Leibniz, Specimen], *De Ipsa natura* 1698 [Leibniz, De ipsa], Response to Nieuwentijt 1695 [Leibniz, Responsio], *Historia et Origo* [Leibniz, Historia] and *Initia rerum mathematicarum* 1715 [Leibniz, Initia].

Although there is a close relation between the treatises on mathematics (mainly the calculus) and physics (mainly the theory of motion), the latter were separately discussed in Chap. 3. This procedure follows the historical development since the exclusively and direct application of the calculus for solving mechanical problems appeared only in the beginning of the 18th century and was due to Varignon, the Bernoulli brothers, Daniel Bernoulli, Euler and d'Alembert. The first comprehensive treatise exclusively based on Leibniz's version of the calculus was written by Euler between 1734 and 1736 [Euler E015/016] and on earlier papers where Euler used the two main representations of the mechanical equation of motion [Euler E069] and the calculus of differences [Euler 1727]. Although Newton invented the calculus two decades before he published the final version of the *Principia*, he did not explicitly made of his new method for the calculation of the relation between fluents and fluxions or, the relation between "the phenomena and the forces". The reason was Newton's aversion to the unification of "arithmetic of variable and geometry" by Descartes and his adherents and the application of the new method to mechanics.[12]

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[11] "Newton had been scooped but even this event [Leibniz's publications on the calculus in 1684 and 1686] did not trigger him to go into print himself; although he did give a hint of his calculus in his *Principia* and John Wallis mentions it in 1693, but the first, main reference to it was in an appendix to Newton's other great book, *Optiks*, which was published in 1704, 20 years after Leibniz's publication." From "Man on the Moon" on the "Automatic for the People CD", 1992: [http://courses.science.fau.edu/∼rjordan/phy1931/NEWTON/newton.htm]

[12] "I have often heard him censure the handling geometrical subjects by algebraic calculations; and his book of Algebra he called by the name of Universal Arithmetic, in opposition to the injudicious title of Geometry, which *Des Cartes* had given to the treatise, wherein he shews, how the geometer may assist his invention by such kind of computations. He frequently praised *Slusius, Barrow* and *Huygens* for not being influenced by the false taste, which then began to prevail. He used to commend the laudable attempt of *Hugo de Omerique* to restore the ancient analysis, and very much esteemed Apollonius's book *De sectione rationis* for giving us a clearer notion of that analysis than we had before. (...) He thought him [Hyugens] the most elegant of any mathematical writer of modern times, and the most just imitator of the ancients. Of their taste, and form of demonstration, Sir ISAAC always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the work of *Des Cartes* and other algebraic writers, before he had considered the elements of *Euclide* with that attention, which so excellent a writer deserves" [Pemberton].

> Men of recent times, eager to add to the discoveries of the Ancients, have united the arithmetic of variable with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output, but the advance is less of a blessing if you look at the complexity of the conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout way quantities which in geometry were designated by the drawing of a single line." [Newton, Math 4:421]

Although Newton condemend the arithemetical method, he made use of the calculus for the discovery of the laws of motion [Newton, Principia, Book II, Sect. II, Lemma II], but presented the results in another frame of reference based on geometry. Similarly to Archimedes who practiced the method of exhaustion in order to *present* the result which had been obtained by application of other methods.[13]

From Newton's statement it follows, that after the invention of the calculus two mathematical formalisms (languages) based either on geometry or arithmetic (algebra) were available for the representation of the relation between bodies, motions and forces. This development had been initiated by Descartes who represented geometrical relations by algebraic expressions. After Newton and Leibniz had presented the corner stones of the new science of motion between 1684[14] and 1687, their contemporaries and followers had to struggle with the problems appearing in mechanics and mathematics to attain the rigour in demonstrations known from geometry and the ancients.[15]

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[13] "Der wesentliche Unterschied gegen die moderne Auffassung besteht darin, daß die Existenz einer Inhaltszahl des Kreises als etwas ganz Selbstverständliches stillschweigend angenommen wird, während die moderne Infinitesimalrechnung auf diese anschauliche Evidenz verzichtet und vielmehr auf Grund des abstrakten Grenzbegriffes die Inhaltszahl als Grenzwert der Maßzahlen eingeschriebener Polygone definiert. (...) Eine von H. Heiberg 1906 entdeckte Schrift des Archimedes zeigt nun in der Tat, daß dieser bei seiner Forschung das Exhaustionsverfahren gar nicht anwandte. Erst nachdem er seine Resultate anderweitig gefunden hatte, bildete er hinterher, um den damaligen Anforderungen an Strenge zu genügen, den Exhaustionsbeweis aus. Zur Entdeckung seiner Sätze benutzte er jedoch eine Methode, die Schwerpunktbetrachtungen [und] den Hebelsatz (...) zu Hilfe nahm (...)" [Klein, Elementarmathematik, p. 225].

[14] "Among his achievements in all areas of learning, Leibniz's contributions to the development of European mathematics stand out as especially influential. His idiosyncratic metaphysics may have won few adherents, but his place in the history of mathematics is sufficiently secure that historians of mathematics speak of the 'Leibnizian school' of analysis and delineate a 'Leibnizian tradition' in mathematics that extends well past the death of its founder. This great reputation rests almost entirely on Leibniz's contributions to the calculus. Whether he is granted the status of inventor or co-inventor, there is no question that Leibniz was instrumental in instituting a new method, and his contributions opened up a vast new field of mathematical research." [Jesseph, Leibniz] Compare also Euler [Euler E212, Preface].

[15] "I recognize (...) that you have written some profound and ingenious things concerning various infinite bodies [de corporibus varie infinitis]. I think that I understand your meaning, and I have often thought about these things, but have not yet dared to pronounce upon them. For perhaps the infinite, such as we conceive it, and the infinitely small, are imaginary, and yet apt for determining real things, just as imaginary roots are customarily supposed to be. These things are among the ideal reasons by which, as it were, things are ruled, although they are not in the parts of matter. For if we admit real lines infinitely small, it follows also that lines are to be admitted which are terminated at either end, but which nevertheless are to our ordinary lines, as an infinite to a finite. Which things being posited, it follows that there is a point in space which can not be reached in an

2.1 Newton's Program for Mechanics

In 1687, Newton summarized the content of his long-lasting investigations of the motion and interaction of bodies in three axioms which form the basis of all developments of mechanics in the following centuries including quantum mechanics. The development of Newton's ideas from the beginning to the final results had been thoroughly analyzed and reconstructed only in the 20th century [Keynes], [Truesdell], [Westfall, Never].[16] In the 19th century, a thorough and comprehensive, but nevertheless critical reconsideration of Newton's basic concepts of space and time and the theory of motion had been performed by Mach [Mach, Mechanik]. Mach accentuated that Newton encountered enormous difficulties in solving the problem to generalize the conceptual basis known from static in order to establish the science of motion.[17] Mach rejected Newton's concept of absolute space and time [Mach, Mechanik], but ignored Leibniz's attempts of a foundation of mechanics and condemned Leibniz for his metaphysical and theological thinking [Mach, Mechanik]. Contrary to the picture drawn by Mach, current investigations of Newton's work reveal the same tight correlation between physics and theology in Leibniz and Newton [Snobelen, Newton].[18] This revised opinion about Newton had already been established by Keynes in the 1930's [Keynes][19] (for current investigations compare Snobelen [Snobelen]).

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assignable time by uniform motion. And it will similarly be required to conceive a time terminated on both sides, which nevertheless is infinite, and even that there can be given a certain kind of eternity (as I may express myself) which is terminated. Or further that something can live so as not to die in any assignable number of years, and nevertheless die at some time. All which things I dare not admit, unless I am compelled by indubitable demonstrations." [Leibniz, GM III, pp. 499–500] Quoted from [Jesseph, Leibniz].

[16] For the present state of art see [http://www.newtonproject.sussex.ac.uk].

[17] "Das Pleonastische, Tautologische, Abundante der Newtonschen Aufstellungen wird übrigens psychologisch verständlich, wenn man sich einen Forscher vorstellt, der, von den ihm geläufigen Vorstellungen der Statik ausgehend, im Begriff ist, die Grundsätze der Dynamik aufzustellen." [Mach, Mechanik]

[18] "Newton's integrated programme for science and religion. The foregoing must not be taken to mean that the influence only flowed from Newton's theology to his natural philosophy. The same considerations that explain this direction of influence also make the reverse direction reasonable. Thus, Newton's methodological approach to the interpretation of prophecy may owe something to his satisfaction with the results of mathematics. It is also clear that Newton's conception of God was in part based on a possibly unconscious desire to create God in his own image. And so in his letters to Bentley Newton spoke of the 'cause' of the solar system being not 'blind & fortuitous, but very well skilled in Mechanicks & Geometry.' Newton's published and unpublished writings demonstrate that his religion interacted with his science at a high level. Newtonian physics cannot be disentangled from Newtonian theology. The lack of firm barriers within Newton's intellectual life suggests that it may even be problematic to speak in terms of 'influence' of one sphere on another. Instead, Newton's lifework evinces one grand project of uncovering God's Truth. Science and religion for Newton were not two distinct programmes, but two aspects of an integrated whole. For Newton, the unity of Truth meant that there was one culture, not two." [Snobelen, Newton]

[19] After reading Newton's manuscripts on alchemy, Keynes stated: "Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the

In 1687, Newton presented the three basic laws of mechanics in the *Principa* [Newton, Principia] after having probed other axiomatic foundations in the 1680's years [Westfall, Never]. The basic laws are formulated for a moving body whose state of uniform motion in the same direction is changed by a moving force impressed upon the body.

> Lex. 1. Corpus omne preservare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum ille mutare.
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> Lex. 2. Mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam, qua vis illa imprimitur.
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> Lex. 3. Actioni contrariam semper & aequalem esse reactionem: sive corporum duorum actiones semper esse aequales et in partes contraria dirigi.
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> Corol. I. Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere, quo latera separatis. [Newton, Principia][20]

Motion is defined by the product of *mass* and *velocity*. Newton introduced motion as an extensive quantity. The motion of a whole is the sum of the motions all parts.[21] Following Galileo, the basic law for natural motion or falling bodies is *independently* of the *mass* and the *shape* of bodies. The motion of projectiles is also geometrically represented by a parabola. Newton made also use of this model, but interpreted the model differently to confirm the conditions for the *preservation of motion* which is only modified by the impact with other bodies.[22] Hence in Newton's interpretation, "uniform motion" means that the *product* of mass and velocity is constant whereas in Galileo's and later interpretations the *velocity* is the only quantity to describe the preservation and the change of the state. Here, the parameter which had been later called "heavy mass" is to be taken into account to obtain Galileo's result. The independence of the mass is correlated with the model where the force is represented by the homogeneous gravitation field. Nevertheless, motion

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last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago." [Keynes]

[20] "LAW I. Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

LAW II. The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

LAW III. To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

COROLLARY I. A body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart." [Newton, Principia (Motte)]

[21] "*The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly*. The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple." [Newton, Principia, Definitions (Motte)]

[22] "PROJECTILES persevere in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in more free spaces, preserve their motions both progressive and circular for a much longer time." [Newton, Principia, Axioms (Motte)] Although Newton formulated the axiom for the "*preservation of state*", the illustration is for the "*preservation of motion*" where the bodies describe *curvilinear* paths like projectiles, tops, planets and comets and included the preservation of momentum.

is always also governed by the inert mass. Later, Euler accentuated the role of inert mass and invented for the first time an operational definition of mass which had been acknowledged by Jammer [Jammer, Mass] (compare Chap. 4). Furthermore, Euler distinguished between internal and external states [Euler E842, § 30].[23]

Newton's program is based on the earlier versions of *De gravitatione* [Newton, De gravitatione] and *De motu* [Newton, De motu][24] where Newton assumed that "force is the causal principle of rest and motion" [Newton, De gravitatione, Def. 5]. The inertia is defined as an "internal force" [Newton, De gravitation, Def. 8]. Newton preserved this definition in the *Principia*. In the 18th century, although Newton's assumption on the inertia was accepted at beginning, Euler, d'Alembert and other authors finally rigorously rejected the concept of a "force of inertia" [Euler E842].[25] In the treatise *De motu*, a preliminary version of the *Principia*, Newton summarized the suppositions for the new theory which had been later essentially modified and only partially preserved. The different stages of Newton's evolution of basic concepts had been analyzed by Westfall [Westfall, Never]. Newton assumed the notions of "impetus" and "conatus" (Definitions 6 and 7) which had been later also treated by Leibniz [Leibniz, Specimen].

In mechanics, Newton developed a method for the solution of two different, but correlated problems which are mechanically modelled by the distinction between *phenomena of motion* and *forces of nature*.[26] The phenomena are described in terms of geometry[27] and are caused by these forces.[28] Newton assumed that either the

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[23] "30. *Man sagt, ein Körper verbleibe in ebendemselben Zustande, wenn derselbe entweder in Ruhe verbleibt oder seine Bewegung nach ebenderselben Richtung mit einerlei Geschwindigkeit fortsetzet*.

Man kann sich in einem Körper einen doppelten Zustand vorstellen, den äusserlichen und den innerlichen. Dieser bestehet in der Art der Theile, aus welchen der Körper bestehet, und ihrer Zusammensetzung selbst; der äusserliche Zustand aber, von welchem allhier allein die Rede ist, besteht in den Verhältnissen des Körpers mit dem Raume." [Euler E842, § 30]

[24] "It is difficult, from our modern Olympian perspective, to understand the mindset of Newton's days. Even the concept of 'velocity' was relatively new at that time. While Newton was inventing mechanics, he was also inventing the very *language* in which it is expressed. And Newton's great nemesis, in all his ruminations, was the concept of infinitesimal. He constantly had to confront Zeno's paradox, and the many apparent contradictions arising therefrom, in all his considerations of fluxions and fluents and quadrature and acceleration." [http://www.ams.org/notices/200311/rev-krantz.pdf]

[25] The rejection of the force of inertia was an essential step in the development of mechanics by Euler since the origin of forces was exclusively related to the interaction of bodies, but not to an internal force which ensures the preservation of state (compare Chap. 4).

[26] Leibniz distinguished between corporeal phenomena and mechanical laws [Leibniz, Specimen, I (13)]. Châtelet translated: "En effet toute la difficulté de la Philosophie paroit consister à trouver les forces qu'employe la nature, par les Phénomenes du mouvement que nous connoissons, & à démontrer ensuite, par là, les autres Phénomens." [Principia, Newton (Châtelet)]

[27] "Lines are described, and by describing are generated, not by any apposition of parts, but by a continual motion of points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, (. . .). These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies." [Newton, Quadrature, (Harris)]

[28] Newton solved the *inverse* problem for planetary motion. The direct problem had been treated by Hermann and Johann Bernoulli. "In 1710, Jean Bernoulli pointed out that Newton had not proved

phenomena or the forces are known. Thus, knowing the phenomena one has to derive the forces and, knowing the forces one has to calculate the geometrical representations in terms of phenomena being caused by the forces [Newton, Principia] (compare Chap. 1).

Leibniz did not essentially modify this program since he also related the *phenomena* to *geometry* and completed geometry by additional principles which are also related to the forces.[29] Following Leibniz, the motion as far as being separated from the forces is necessarily only *respective* or *local* motion [Leibniz, Specimen, I (2)]. Leibniz specified the "phenomena" to the "corporeal phenomena" which is the basis of his relational approach to time and space. As it follows from the title *Philosophiae naturalis principia mathematica*, Newton invented his program for mechanics in order to study mathematically and mechanically the relation between *phenomena* and *forces of nature*.

This subdivision into phenomena and forces results simultaneously in the definition of two main problems. Although the problems are of different type, there is a common methodological basis.[30] Later in 1736, Euler removed all metaphysical implications introduced by Newton and Leibniz and formulated the problem for the relative motion of bodies constituting a world which consists only of bodies and space.[31] As a consequence, Euler redefined the relation between phenomena and

Kepler's law of ellipses but only its converse and did so himself using calculus, solving 'the general problem by reducing it to the same integral that is used to solve it today' (Park 1990:416). (...) In 1742, Jean Bernoulli, in *Opera omnia*, proved that the orbits of objects bound by the inverse square force are conic sections." [Time Line] "Hermann was the first to study what, today, we call the direct Kepler problem: namely, given the central force inverse square law, determine the orbit. In spite of the pioneering excellence of this analysis, including the proof that in such a force-field all orbits are conic sections, it was immediately superseded by the more comprehensive treatment of Johann I Bernoulli. Although Hermann had priority in tackling the problem, he had not obtained the complete solution." [http://www.fyma.ucl.ac.be/∼gaino/Bernoulli/JacobHermann.html] "Proposition 1 of Book 1 of Newton's *Principia* (1687), which states that Kepler's area law holds for any central force, plays a fundamental role in the study of central force motion. Newton's geometric proof of this proposition is based on an intuitive theory of limits. In 1716–1717 the Swiss mathematician, Jakob Hermann, gave a proof of Proposition 1 based on infinitesimals. The present paper discusses both Newton's and Hermann's solutions. A comparison of the two gives us an insight into an episode of the process that led from the geometric style of Newton's *Principia* to the analytic style of Euler's *Mechanica* (1736)." [Guicciardini, Hermann]

[29] Leibniz called the "forces of nature" substances or "simple things" who are the true causes of the phenomena as the Newtonian forces of Nature are. "I am believe of the opinion that, to speak exactly, there is no need of extended substance. (...) True substances are only simple substances or what I call 'monads'. And I believe that there are only monads in nature, the rest being phenomena of them." [Leibniz, Letter to Dangicourt]

[30] Leibniz based mechanics upon the same distinction between corporeal phenomena and superordinate principles. Refuting the attempts of Henry Moore and Aristotle to relate the phenomena to a fundamental principle, Leibniz stated that the *corporeal phenomena* can be deduced from *mechanical causes*, but the *mechanical laws* have to be deduced from *superordinate principles*: "Optimum meo judicio temperamentum est, quo pietati et scientiae satisfit, ut omnia quidem phaenomena corporea a causis efficientibus mechanicis peti posse agnoscamus; sed ipsas leges mechanicas in universum a superioribus rationibus derivari intelligamus; atque ita causa efficiente altiore tantum in generalibus et remotis constituentis utamur." [Leibniz, Specimen, I (13)]

[31] "Hier werden diejenigen Veränderungen mit Fleiss ausgeschlossen, welche unmittelbar von Gott oder einem Geiste hervorgebracht werden. Wenn wir also in der Welt nichts als Körper betrachten,

forces in terms of the *paths* the bodies describe and the *forces* the bodies generate by their interaction. Euler's program is commensurable with the contemporary terminology (compare Chap. 4). Preserving the subdivision into internal and external principles, the basic intention is to define *algorithms* for reckoning of the paths based on the application of the calculus and mechanical principles which are as reliable as the laws of geometry: (N1) Calculate the paths if the forces are given, (N2) Calculate the forces if the paths are given (compare Chap. 4).[32]

Newton and Leibniz developed complementary programs accentuating the change of motion and the presevation of "living forces", i.e. the preservation of the quantity of motion, respecitvely. Hence, Leibniz mechanics includes implicitly a "hidden force model" whereas Newton's mechanics implies, for some special kinds of forces, a "hidden energy model" or "conservation of living forces model" represented by the expressions for the change of motion and the preservation of living forces.

> (...) nec plus minusve potentia in effectu quam in causa contineatur. [Leibniz, Specimen, I (11)]

Although Leibniz implicitly made use of the *inertia* of bodies and the invariance of the inert mass,[33] he added that this law is not derived from the notion of *mass*, but it has to be traced back to somewhat other, i.e. the *inherent* forces the bodies:

> Quae lex cum non derivetur ex notione molis, necesse est consequi eam ex alia re, quae corporibus insit, nempe ex ipsa vi, quae scilicet eandem semper quantitatem sui tuetur, licet a diversis corporibus exerceatur. [Leibniz, Specimen, I (11)]

This additional principle cannot be derived from mathematics, but it is only comprehensible to the reason and, consequently, it has to be formulated in terms of metaphysics such as cause and effect, action and suffering [Leibniz, Specimen, I (11)].

In the Leibnizian formula, there are no *explicit expressions* for Newtonian "impressed moving forces"[34] whereas in Newton's law there is no explicit expression for energy or "living forces". The step beyond the Newtonian program is performed

so ist klar, dass ein jeder Körper so lange in seinem Zustande verbleiben muss, als sich von aussen keine Ursache ereignet, welche vermögend ist, in demselben eine Veränderung zu wirken." [Euler E842, § 49] (compare Chap. 4, Section *Euler's world models*).

[32] "C'est aussi à quoi aboutissent toutes les recherches de la Mécanique, où l'on s'applique principalement à deux choses: (i) l'une, les forces qui agissent sur une corps étant données, déterminer le changement qui doit être produit dans son mouvement; (ii) l'autre, de trouver les forces mêmes, lorsque les changements, qui arrivent aux corps dans leur état, sont connus." [Euler E181, § 10]

[33] Leibniz claimed that the "earlier hypothesis", i.e. on inertia and impenetrability, on the bodies is "incomplete": "(...) vidi in quo consisteret systematica rerum explication, animadvertique hypothesin illam priorem notionis corporeae non esse completam; et cum aliis argumentis, tum etiam hoc ipso comprobari, quod in corpore praeter magnitudinem et impenetrabilitatem poni debeat aliquid, unde virium consideratio oriatur; (...)." [Leibniz, Specimen, I (11)]

[34] As reviewed by Voltaire, Newton claimed: "Let us listen to Newton and the experience and stop this metaphysical disputation. The motion, Newton said, is generated (produced) and lost. But, due to the tenacity of the fluids and the elasticity of the solids, is more motion lost as it is renewed in nature." [Voltaire, Élémens, Chap. IX]

by the assumption of *conservation law* for living forces which is related to the rejection of perpetual motion.[35] The assumption of living forces had been later acknowledged by Euler, d'Alembert, the Bernoullis and Châtelet [Châtelet, Institutions].[36]

2.2 Newton and Leibniz on Time, Space, Place and Motion

Leibniz's relational definition of time and space [Leibniz, Initia] is known to be essentially different and in strong contrast to Newton's theory of absolute time and absolute space [Newton, Principia]. In 19th century, Mach's well-known criticism ends up in the complete rejection of these basic notions of Newton's theory. However, Mach did not mention the contribution of Leibniz as a predecessor in the criticism of that part of Newton's theory. He also did not discuss the relation between absolute and relative times and spaces which have been introduced in the *Principia*. After the advent of Einstein's theory of relativity it was extremely difficult to reconstruct the sophisticated details of the Newton – Leibniz controversy since people declared Leibniz to be the winner. In 1924 Reichenbach stated that the success of Newton's *Principia* has hampered the development of mechanics for 200 years [Reichenbach].

The comparison between the ranking order of time and space found in the title of Reichenbach's book on *The philosophy of space and time* and that of the same concepts in Newton's *Principia* and Leibniz's *Initia* demonstrates the difference in the weighting in 17th and 20th centuries, however, it reveals also an astonishing agreement between Newton's *Principia* and Leibniz's *Initia*. Despite their controversy on the definition of time and space Newton and Leibniz agreed almost perfectly in the *order* of introduction of the concepts of time, space and motion. This order had been only called into question by Euler who based his considerations on the definition of *rest* and *motion* related to the place [Euler E015/016, § 1], [Euler E289, §§ 1–10] and the axiom of the uniform motion of bodies which is the basis for the definition of a *quantitative* relation between time and space intervals [Euler E149].

The difference and opposition between Newton and Leibniz are not due to the set of basic notions of time, space, place and motion, but due to the *logical structure* of

[35] The rejection of perpetual motion follows from Leibniz's statement: "(...) nec plus minusve potentiae in effectu quam in causa contineatur" [Leibniz, *Specimen*, I (11)], there is neither more nor less potency in the effect than in the cause, i.e. the conclusion (not the assumption) is that the only remaining case follows as the equality of cause and effect or the conservation of the potency. Mathematically, there are three cases, (i) effect is *less* than cause, (ii) effect is *larger* than cause and (iii) effects is *equal* to the cause. Newton did not make a choice, but stated only that there is a relation between "effect" and "cause", i.e. change in motion and impressed moving force, respectively. "Mutationem motus *proportionalem* esse vi motrici impressae, (...)." [Newton, Principia, Axioms] (compare also Chap. 7).

[36] In the translation of Newton's Principia, Châtelet replaced "forces of Nature" ("vires Naturae") with "forces which are employed by nature". "En effet toute la difficulté de la Philosophie paroit consister à trouver les forces qu'employe la nature, par les Phénomenes du mouvement que nous connoissons, & à démontrer ensuite, par là, les autres Phénomenes." [Newton, Principia (Châtelet)]

the assumed statements and the logical status of the notions. The basic difference for all notions is whether the objects are analyzed concerning their properties or their relations. Obviously, Newton's theory of absolute time is not based upon a relational concept of time since it is considered "without regard to anything external" [Newton, Principia, Definitions]. However, Newton completed the *non-relational* concept of *absolute* time (space and motion) with a *relational* concept of relative time, space, place and motion. All basic expressions appear doubly. No decision was made in favour of one of the two basic sets. Moreover, no decision can be made since the absolute quantities are related to mathematics and the relative quantities are related to measurement. As a consequence, all followers of Newton, his adherents and opponents, had to handle this problem and they did it differently. The same statement is true for Leibniz and the Leibniz-Wolffian school since the theory of Leibniz is based on a similar ambiguity, now related to forces. Also Euler developed the mechanics twofold and always compared the frame of relative motion to the frame of absolute motion. However, in contrast to most of his contemporaries who intended to merge both the topics, Euler made a clear decision on favor of *relative* motion (compare Chaps. 4 and 6). Instead of the classification according to absolute and relative concepts, Euler based the theory on two kinds of principles, *internal* and *external* principles of *motion*, where the internal principle are related to the isolated non-interacting body whereas the external principles are related to the interaction of bodies. Euler's mechanics is a consequent and true *relational theory of motion*, therefore, Euler may be considered as a predecessor of Einstein.

2.2.1 Newton and Leibniz on Time and Space

In the 20th century, the essential difference between the absolute and relative time and space had been accentuated [Reichenbach], [Reicehnbach, Space and Time].[37] Newton's and Leibniz's approaches to establish a general frame for time, space, place and motion are very similar and largely coinciding as far as their specification to absolute and relative quantities is not considered. Newton's frame reads as follows.

[37] "Der Briefwechsel liest sich ähnlich wie eine moderne Diskussion über Relativitätstheorie; der Relativist sucht vergeblich einen Gegner zu überzeugen, der so in der absolutistischen Vorstellung befangen ist, dass er gar nicht merkt, wie sehr seine Argumente die Lehre voraussetzen, die sie erst beweisen wollen, und wie die vermeintlichen Widersprüche, die er dem Relativisten nachweisen will, eben nur auf einer ständigen Unterschiebung der absolutistischen Auffassung beruhen. (...) Es ist jedoch das seltsame Schicksal Newtons, dass er, der mit seinen physikalischen Entdeckungen die positive Wissenschaft reich befruchtete, zugleich die Entwicklung der begrifflichen Grundlagen dieser Wissenschaft weitgehend gehemmt hat. So fruchtbar seine optischen Entdeckungen waren – mit seiner Emissionstheorie des Lichtes hat er die Anerkennung der Wellentheorie (...) um ein Jahrhundert zurückgehalten. Und so weittragend seine Entdeckung des Gravitationsgesetzes war – die Analyse des Raum- Zeitproblems wurde durch seine Mechanik um mehr als zwei Jahrhunderte aufgehalten, nachdem sein Zeitgenosse Leibniz bereits wesentlich tiefere Einsichten in die Natur von Raum und Zeit gehabt hatte." [Reichenbach]

> I. Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.
>
> II. Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies. I say, a part of space; not a situation nor the external surface of a body.
>
> III. Place is a part of space which a body takes up, and is according to the space, either absolute or relative.
>
> IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. [Newton, Principia (Motte)]. (comment (i) measurement,[38] (ii) fluxions[39] and (iii) absolute and relative[40])

In contrast to Newton who related only two different states, i.e. the states of "rest and uniform motion", to *bodies*, Leibniz generalized the procedure assuming a manifold of "states of things" which are either compatible or incompatible [Leibniz, Initia] without specifying these states as different states of bodies.[41]

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[38] Following Newton, the difference to the "*mathematical time*" is that the "*mechanically defined time*", although being also numerically represented by quantities, are *discrete* quantities. Beside the difference between mathematical and mechanical time Newton assumed implicitly the complementarily between "continual and discrete" mechanical quantities. This relation is excluded from mathematics since the geometrical line is not composed of parts. "I don't here consider Mathematical Quantities as composed of parts extremely small, but as generated by a continual motion." [Newton, Quadrature (Harris)] The "mechanical time" has to be related to measurement. The measured quantities are necessarily discrete, *one* hour, *one* day, *one* month, *one* year and so on, whose relation can be expressed in terms of *finite numbers* being either *integers* or *rational numbers*. Newton excluded *indivisible* time elements and indivisible space elements. The result is equivalent to and even stronger (more restrictive and more sophisticated) than Leibniz's assumption. Leibniz excluded the "existence" of parts whereas Newton excluded any "apposition of parts" ("not by any apposition of parts"). Thus, the "apposition of parts" is declared to be forbidden whereas the "decomposition into parts" is assumed to be allowed for the purpose of measurement.

[39] Mathematically, the "time" defined by Newton is related to the continuum and, mechanically, to a continual motion. The "time" does not consist of parts, neither of divisible nor indivisible parts. Leibniz's theory of time and motion is based on a similar assumption on the "parts". "Nam motus (perinde ac tempus) nunquam existit, si rem ad $\alpha\kappa\rho\iota\beta\varepsilon\iota\alpha\nu$ revoces, quia nunquam totus existit, quando partes coexistentes non habet." [Leibniz, Specimen, I (1)], but Leibniz accentuated the relation between "parts and whole" instead of the "generation" of quantities [Newton, Method of Fluxions].

[40] Euler argued in favour of Leibniz. "En ergo realem quietis definitionem nullis ideis vagis seu imaginariis implicatam, quae autem coniuncta est cum idea cuiuspiam corporis, cuius respectu punctum *O* quiescere dicitur; neque patet, quid sit quies absolute sic dicta separata a talis corporis notione" [Euler E289, § 8]. The absolute rest had been discussed *after* the relative rest (and motion) had been introduced and defined. In the *Mechanica*, Euler discussed absolute and relative rest (and motion) on an equal footing.

[41] Leibniz's goal is not only the foundation of physics, but also to find a method which answers the purpose of a foundation of mathematics: "(...) esse artem quandam Analyticam Mathematica ampliorem, ex qua Mathematica scientia pulcherrimas quasque sua Methodus mutuatur." [Leibniz, Initia, p. 353] This procedure corresponds to the difference in the foundation of the calculus by Newton and Leibniz where Newton preferred a *universal flux in time* and Leibniz time favoured *geometrical* and *algebraic methods* being alien to Newton's assumptions (compare

Referring to the primarily stipulated and distinguished states of the things, Leibniz defined *time* and *space* in one the same procedure simultaneously, i.e. as *correlated logical* operations[42] with respect to states, as expressing different relations between these states which are described as *different orders* of these things. Newton defined time as a permanent and uniform *flux* and *space* as an existing invariant thing.

In spite of these differences, Leibniz introduced the concepts in the same order as Newton. This almost perfect matching is remarkable since Leibniz has written his treatise after Newton has published the *Principia*. As Newton did, Leibniz defined firstly time (and duration), secondly, space. Time is the order of the non-coexisting things.

> Si plures ponantur existere rerum status, nihil oppositum involventes, dicentur existere simul. Et ideo quicquid existit alteri existenti aut simul est aut prius aut posterius.
>
> Si eorum quae non sunt simul unum rationem alterius involvat, illud prius, hoc posterius habetur.
>
> (I′) Tempus est ordo existendi eorum quae non sunt simul.
> (II′) Spatium est ordo coexistendi seu ordo existendi inter ea quae sunt simul.
> (III′) *Situs* est coexistentia modus.
> (IV′) *Motus* est mutatio situs. [Leibniz, Initia]

The only modification is that Leibniz replaced the place (*locus* (III)) with the situation (*situs* (III′)). In contrats to Newton's procedure,[43] an explicit definition of *rest* is missing. It is, however, included in the definition of situs. Beside the simultaneity related to the *states* of the things, which are either simultaneously or non-simultaneously, Leibniz defined for the measurement a special type of simultaneity, called *compraesentia*, using the notion of situs. *Situs* is a modus of coexisting things comprising quantity and quality. The quantity can be only known from the comparison of different things which form for a certain time a system or, are in a stable configuration, e.g. a body and the unit for measuring the length of the body. Leibniz called this *simultaneous existence* or to be *simultaneous present* (compraesentia) so that they are simultaneously *perceived* (seu perceptione simultanea) [Leibniz, Initia][44] by an observer. The procedure defines the conditions for measurement if one of the coexistent things which are compared to each other is chosen to be the gauge (unit).[45] However, Leibniz is aware of the problem to define spatial

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Chap. 3). Although the resulting operations turned out to be independent of the foundation, the inherent advantage of the Leibnizian method is that time and space as variables may be treated on an equal footing (compare Euler's derivation of the two equations of motions [Euler E015/016, §§ 131–155]).

[42] Coexistence (space): "State1 AND State2", Succession (time): "EITHER State1 OR State2" (State1 and State2 are abbreviations for the statement "The system is in State I", $\mathrm{I} = 1, 2$).

[43] "But real, absolute rest, is the continuance of the body in the same part of that immovable space, (...)." [Newton, Principa. Axioms]

[44] "*Quantitas* seu Magnitudo est, quod in rebus sola compraesentia (seu perceptione simultanea) cognosci potest." [Leibniz, Initia]

[45] Euler discussed the same procedure for the definition of units [Euler E289, § 6]. Having the unit defined the result of the measurement can be expressed in terms of numbers. This procedure is applicable to all physical quantities. The problem of the invariance of units had been already emphasized by Leibniz [Leibniz, Nouveaux Essais, Vol. I, Book II, Chap. XIII, § 4].

and temporal order by the same principle of "compraesentia" since two perceptions appearing in a temporal order can only be compared to each other if there is in fact a *transition* but really neither an *annihilation* of the first nor a *creation* of the second thing [Leibniz, Initia].[46] According to these principles, Leibniz defined the relative translation of a body A with respect to other bodies C, D, E and G whose relative positions are not changed by the translation[47] of the body A. The translation is not related to a definite time interval but is only described geometrically. Euler renewed this model for describing the position, or situation called *situs* [Euler E289, § 3]. As Leibniz, Euler considered relative translations, but extended the analysis by inclusion of relative *motion* [Euler E842][48] thereby considering translations which are performed in a finite or infinitesimal time interval.

Therefore, the distinction between basic quantities is now made predominantly between *absolute and relative motions* rather than between absolute and relative *space and time* (compare Chap. 6). Euler's approach allows for a new consideration of the whole problem since it contains implicitly the question whether or not the motion, described by the *velocity* of the body, is limited in its magnitude. In the case of time and space it was clear for Newton and Leibniz that both the quantities are not limited but are infinite. These assumptions have been introduced axiomatically without reference to motion. In contrast, the statement on the unlimited magnitude of velocity has to be deduced within a certain model. Leibniz considered the rotation of a disk and concluded that the velocity can be increased without limitation by the increase of the distance from the center of disk.

Leibniz compared the order between bodies and the order between numbers. The order is changed by motion. What about the rest?

> Et comme les corps passent d'un endroit de l'espace à l'autre [from one place to another place], c'est à dire qu'ils changent l'ordre entr'eux, les choses aussi passent d'un endroit de l'ordre ou d'un nombre à l'autre, lorsque par exemple le premier devient le second et le second devient le troisième etc. (...) En effect le temps et le lieu ne sont que des espèces d'ordre, et dans ces ordres la place vacante (qui s'appelle vuide à l'egard de l'espace) s'il y en avoit, marqueroit la possibilité seulement de ce qui manque avec son rapport à l'actuel. [Leibniz, Nouveaux Essais, Vol. I, Book II, Chap. IV, § 5, p. 138][49]

Leibniz assumed free and occupied places within the orders, but no empty space. What about vacant position in time? Leibniz did only treat time and space as orders on an equal footing, but did not demonstrate the equivalence for "empty places" in space and time.

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[46] "Coexistere autem cognoscimus non ea tantum quae simul percipiuntur, sed etiam quae successive percipimus, modo ponatur durante transitu a perceptione unius ad perceptionem alterius aut non interiisse prius, aut non natum esse posterius." [Leibniz, Initia, p. 368]

[47] Leibniz used a modified version of the model of the stadium of Zeno and Aristotle [Leibniz, Clarke]. In the original version the bodies represented by chariots are *moving* a certain distance in a certain time interval. Leibniz discussed a spatial *translation* instead of motion.

[48] In 20th century, Reichenbach interpreted Leibniz's theory of space and time as a relational theory which contrasted the notions of Newton [Reichenbach]. However, Reichenbach's interpretation of the Leibnizian theory is misleading since he identified the Leibnizian theory of relative *position* with the Einsteinian theory of relative *motion*.

[49] "Indeed, time and positions (lieu) are species of orders, and within these orders is a vacant place (which is called empty with respect to the space) if it exists, denoted the mere possibility [not the reality] of that what is missing with respect to the actual." [Leibniz, Nouveaux Essais]

2.2.2 Order and Quantification

The main objection of Clarke against Leibniz's arguments is the missing possibility to relate quantities or numbers to the order. Leibniz distinguished two kinds of order, first those of coexisting things and, second those of non-coexisting things. Following Leibniz two time intervals of different or equal lengths may be considered. Although the magnitude of two time intervals Δt_1 and Δt_2 may be given by comparison as either $\Delta t_1 < \Delta t_2$ or $\Delta t_1 > \Delta t_2$, the relations "before" and "after" are not determinate by these inequalities as long as a common frame of references had not been introduced. The "past" and the "present" year may be of arbitrary duration.

> Itaque quae anno praeterito et praesente facta sunt negamus esse simul, involvit enim oppositos ejusdem rei status. [Leibniz, Initia]

Newton obtained automatically a quantification of time and space (compare below the comment by Clarke). Hence, the shortage of Leibniz's theory is caused by the investigation of relative *translations* instead of relative *motions* of bodies in the same frame.

> 47. I will here show, how men come to form to themselves the notion of space. They consider that many things exist at once and they observe in them a certain order of co-existence, according to which the relation of one thing to another is more or less simple. This order is their *situation* or distance. When it happens that one of those co-existent things changes its relation to a multitude of others, which do not change their relation among themselves; and that another thing, newly come, acquires the same relation to the others, as the former had; we then say, it is come into the place of the former; and this change, we call a motion in that body, where in is the immediate cause of the change.
>
> And, to give a kind of a definition: *place* is that, which we say is the same to A and, to B, when the relation of the co-existence of B, with C, E, F, G etc. agrees perfectly with the relation of the co-existence, which A had with the same C, E, F, G, etc. (...). Lastly, *space* is that, which results from places taken together. [Leibniz Clarke (Alexander), Leibniz's 5th Letter]

This consideration is consequent as long as the change of distances and the spatial relations are considered to be independently of time. The velocity is indeterminate, i.e. the velocities of the bodies A and B are also indeterminate. However, in case of *two bodies* which are either resting or moving *relatively to each other* the velocity is not indeterminate. In the first case, the velocity is zero, in the second case it has a definite finite value.

The objection of Clarke [Leibniz Clarke] was not only directed against Leibniz's assumption of ordering, but it has been also reinforced by the true argument that time and space cannot be considered as mere orders.[50]

> Further: Space and Time are Quantities: which Situation and Order are not. [Leibniz Clarke (Alexander), 3rd Letter to Leibniz][51]

[50] This argument had been renewed by Euler [Euler E149, §§ 20–21]. Euler claimed that space and time are more than mere orders because the *equal* distances in *equal* time a body is travelling cannot be explained by a mere order of succession and coexistence.

[51] The foundation of the criticism of Clarke is due to Newton's basic assumptions on the generation of lines, surfaces and solids by a continual motion which is closely related to the foundation of the

Therefore, Leibniz's criticism of Newton's absolute space is as well justified as Clarke's criticism of the lack of quantities in Leibniz's relational theory. This point of view has been later stressed also by Euler. Finally, both the aspects have been brought together only by Einstein due to the introduction of light velocity into mechanics. Additionally, the assumption of an upper limit for all kinds of motion results in an order between the measurements performed by different observers (compare Chap. 6). The order can be also expressed in terms of Lorentz transformation. Moreover, assuming such ordering relations, it follows that the difference between Galileo and Lorentz transformations results only from the order in the relative motion of bodies which is established without any reference to electrodynamics.

After the invention of calculus, the notions of order and quantification were considerable modified and their mechanical meaning was expanded due to the distinction between finite and infinitesimal quantities or, in terms of mechanics, the motion of bodies with finite velocities and those mechanical quantities being related to the "beginning" and the "end" of motion. Before the invention of "infinitely little quantity" by Newton (compare Chap. 3), the beginning and the end of motion had been only qualitatively discussed by the construction of geometrical models of the continuum [Leibniz, Hypothesis] or by the construction of models for motion by transcreation [Leibniz, Pacidius].[52] Newton decided to concentrate the study of motion to the *very beginning* and the *end* of motion or the *nascent* and *evanescent* motion and to describe these special forms by the first and last ratios of fluents and fluxions (compare Chap. 3).

2.2.3 The Very Beginning of Motion

Following Newton, the very beginning of motion can be described in terms of nascent (or generated or produced [Newton, Principia, Book II, Lemma II, p. 256])

calculus (compare Chap. 3). The ordering is an inherent property of the method of generation since the lines, surface and solids are not generated at an *instant* (or, using Leibniz's terminology "tout d'un coup" [Leibniz, Monadology, § 6]), but an emerging and evolving continuous process whose *beginning* is described in terms of an infinitesimal quantity [Newton, Method of Fluxion]. The beginning cannot be separated from the continuation and a line does not consist of infinitesimal parts. Therefore, the continual motion is not described in terms of an ordered series of elementary steps, represented by "time intervals" o, $2\,o$, $3\,o$, … and the independent variable is not represented by an arithmetical series (compare Chap. 3).

[52] "From among his early attempts on the continuum problem I distinguish four distinct phases in his interpretation of infinitesimals: (i) (1669) the continuum consists of assignable points separated by unassignable gaps; (ii) (1670–71) the continuum is composed of an infinity of indivisible points, or parts smaller than any assignable, with no gaps between them; (iii) (1672–75) a continuous line is composed not of points but of infinitely many infinitesimal ones, each of which is divisible and proportional to a generating motion at an instant (conatus); (iv) (1676 onward) infinitesimals are fictitious entities, which may be used as *compendia loquendi* to abbreviate mathematical reasonings." [Arthur, Fictions] Models of such type had been excluded by Newton who postulated the generation of paths by a continual flux [Newton, Quadrature (Harris)], i.e. a flux which is neither interrupted by gaps of finite extension nor by gaps having no extension.

quantities since this stage represents something in between rest and traversing a finite distance. The evanescent or disappearing quantities are suitable to describe the complementary process. Following Newton, both the processes are analytically represented by the first and ultimate ratios of the augments of fluents in time [Newton, Quadrature, Harris],[53] [Newton, Principia, Book I, Sect. I]. The increments of velocity and the errors are of infinitesimal magnitude. Although these increments of velocity cannot be measured, they are not merely fictitious quantities, but represent the change of the state of the body in the frame of a thought experiment.

> Cor. 1. (...) that the errors of bodies describing similar parts of similar figures proportional times, are nearly in the duplicate ratio of the times in which they are generated, if so be these errors are generated by any equal forces similarly applied to bodies, and measured by the distances of the bodies from those places of similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times.
>
> Cor. 2. But the errors ... are as the forces and the squares of the times conjunctly. ($s_{\text{begin}}^{\text{errors}} \sim K \cdot t_{\text{begin}}^2$) (which is equivalent to $\Delta\Delta s \sim K\Delta t^2$).[54]
>
> Cor. 3. The same thing is to be understood of any space whatsoever described by bodies urged with different forces. All which, in the very beginning of motion, are as the forces and the squares of the times conjunctly.
>
> Cor. 4. And therefore the forces are as the spaces described in the very beginning of the motion directly, and the squares of the time inversely. ($K \sim s_{\text{begin}}^{\text{errors}}/t_{\text{begin}}^2$).
>
> Cor. 5. And the squares of the times are as the spaces describ'd directly and the forces inversely. ($t_{\text{begin}}^2 \sim s_{\text{begin}}^{\text{errors}}/K$). [Newton, Principia, Book I, Sect. I, Lemma IX][55]

Following Galileo and representing the "change in motion" in terms of finite quantities (compare Chap. 1), i.e. finite increments of motion in dependence on finite increments of time, the relations are given as follows where the *increments of velocity* and the *increments of the increments* of the path $\Delta(mv) \sim K$ or $m\Delta v \sim K \cdot \Delta t$ and $\Delta\Delta s \sim K\Delta t^2$ are proportional to the increment of time and the square of the increment of time, respectively. Following Newton, motion is quantitatively described by the product of mass and velocity. Newton called the increment $\Delta(mv)$ "change of motion" and described the change of the state by this quantity instead of making use of the "change of velocity" (for Euler's procedure compare Chap. 4).[56] However, analyzing the general relations between "errors, time and forces", the previously defined *change in motion* in terms of $\Delta(mv)$ had not been explicitly included.

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[53] "Now let those Augments vanish and their ultimate Ratio will be the Ratio of *1* to nx^{n-1}." [Newton, Quadrature (Harris)] The ratio is made of finite quantities also it had been previously represented in terms of infinitesimal quantities. Here, Berkeley found an inconsistency in the foundation [Berkeley, Analyst] (compare Chap. 3).

[54] Comments in brackets by D.S.

[55] Later in the 19th century, only the relation derived in Corol. 2 had been accepted and called differential quotient $d^2s/dt^2 = K/m$. All other relations discussed by Newton had been either ignored or rejected. Obviously, although Newton later argued against differentials [Newton (Collins), Commercium], the assumed relations can be readily interpreted as those being valid for differentials.

[56] "30. Man sagt, ein Körper verbleibe in ebendemselben Zustande, wenn derselbe entweder in Ruhe verbleibt oder seine Bewegung nach ebenderselben Richtung mit einerlei Geschwindigkeit fortsetzet." [Euler E842, § 30]

On the contrary, the analysis is performed for the deviation of the *positions* the body arrives in *uniform motion* from those which are generated due the presences of forces (compare above Cor. 1). The positions and the errors are defined geometrically or "measured by the distances of the bodies from those places of similar figures". The *geometric* frame of reference is formed by the path the body describes in uniform motion. As a consequence, the analysis cannot be efficiently transferred to the *change of rest* since an appropriate frame of reference is missing. Analytically, the frame of reference is formed by the equations $v = 0$ or $v = \text{const}$. Then, the change of the state is always represented by the increment of velocity Δv independently of the initial state of the body being either the state of rest or the state of uniform motion, i.e. $v = 0$ or $v = \text{const}$, respectively (compare Chap. 4). Although the result is independent of the *velocity* of the body [Euler E015/016, § 131], the *magnitude* of the change depends on the inertia or the mass of the body which is even the missing parameter in the relation between errors, times and forces. Uniform motion is *mass independent* whereas the *change* of uniform motion is *not* mass independent. Therefore, the true values of errors are to be obtained by comparison of a mass independent and mass dependent trajectory described by the body.

Hence, there are complementary approaches, (i) Newton presented an explicit relation for the *change in motion* whereas the conservation of living forces was hidden, whereas Leibniz, on the contrary, (ii) presented explicitly the *conservation of living forces* whereas the change in motion by derivative forces was hidden.

Following Leibniz, the efficiency of the action of forces may be characterized by the magnitude of the change of velocity being either a maximum or a minimum. Obviously, the *most efficient* change is the change of the state of rest since there is a transition from rest to motion. Then, the very beginning of motion is described by the equation $\mathrm{d}v = \mathrm{d}v_{\max} = (K/m)\mathrm{d}t$. Consequently, the *most inefficient* change of rest is given by the relation $\mathrm{d}v = \mathrm{d}v_{\min} = (K/m)\mathrm{d}t$ where, obviously, the lower limit is represented by $\mathrm{d}v = \mathrm{d}v_{\min} = 0$, i.e. no change at all. This result can be obtained (i) either for $K = 0$ which is the trivial case or (ii) for $K \neq 0$ and $m \to \infty$ which is the non-trivial case. Following Eule and assuming constant forces of finite magnitude $K_{\text{finite}} = \text{const}$, it is impossible that there is no change of the state [Euler E015/016], [Euler E842], [Euler E289]. The change of the state is described by the increment of velocity $\mathrm{d}v$ or, following Newton, "the errors *dds* in the very beginning of motion". Euler claimed that, according to the principles of least action, the change of velocity is generated by the forces which are the least those ones to avoid the penetration of bodies [Euler E343, Lettre LXXVIII] (compare Chap. 4).

2.2.4 Polygon and Circle: Periodic Motion

Newton analyzed the motion of a body under the influence of external forces in a model based on periodic motion performed by the body along the sides of a polygon inscribed into a circle (see Fig. 2.1). The model system is composed of physical and non-physical components. The body is assumed to move (i) uniformly along the sides of the inscribed polygon, then, arriving at the intersection point of the side

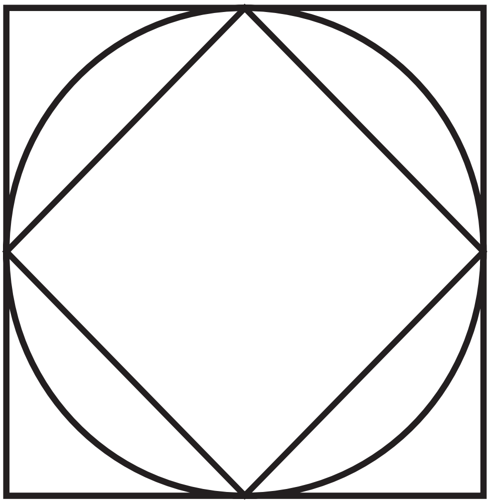

Fig. 2.1 Newton's geometrical model for the relation between the change of velocity caused by forces
Newton's derivation of the change of momentum of a moving body due to a central force [Westfall, Never]

and the circle (ii) the body is reflected at the side of the circumscribed polygon and (iii) continues the uniform motion with the same magnitude of velocity along the consecutive side as long as (iv) it returned after a finite time to its initial position.[57] Although the model comprises non-physical elements like the reflection at the circumscribed polygon, Newton obtained the correct results by an appropriate analytical representation of periodic motion [Westfall, Never, p. 149]. Moreover, Newton made use of the general relation between curved and straight lines represented by the relation between polygon and circle in a plane. This model had been later also use for the representation of the relation between finite and infinite quantities as well as the notion of limit. Lagrange referred to ancient science to establish an appropriate notion of limit. It is possible to approach to this quantity as close as possible, but never attain it or pass through the limit, in the present example from the inscribed to the circumscribed polygon and vice versa.[58] Later in 1695, the critics of the calculus also referred to the model of polygon and curved lines [Nieuwentijt, Analysis],[59] [Weissenborn].

[57] Periodic motions of such type had been analyzed by Leibniz [Leibniz, Brevis] and later by Helmholtz who referred to Leibniz [Helmholtz, Vorlesungen] (compare Chap. 7).

[58] "Les véritables limites, suivant les notions des anciens, son des quantités qu'on ne peut passer, quoiqu'on puisse s'en approcher aussi près que l'on veut; telle est, par exemple, le circonférence du cercle à l'égard des polygones inscrit et circonscrit, parce que, quelque grand que devienne le nombre des côtes, jamais le polygone intérieur ne sortira du cercle, ni l'extérieur n'y entrera." [Lagrange, Fonction]

[59] Nieuwentijt B (1695) Analysis infinitorum seu curvilineorum proprietates ex polygonorum natura deductae. Wolters, Amsterdam.

2.3 Leibniz's Program for Mechanics

In 1695, Leibniz commented on the early version of the theory of motion published in the treatise *Theoria motus abstracti*. "Mihi adhuc juveni, et corporis naturam cum *Democrito* et hujus ea in re sectatoribus *Gassendo* et *Cartesio*, in sola massa inerte tunc constituenti, (...)." [Leibniz, Specimen I (10)]. Later, Leibniz did not explain the conservation of state as Descartes and Newton by inertia. As a result, instead of simplyfing and generalizing the theory, Leibniz was forced to introduce a variety of different forces [Leibniz, Brevis], [Leibniz, Specimen] and run in trouble to define analytically the relation between forces. Nevertheless as in Newton's mechanics, the *increments* or *decrements* of velocity also played a crucial role in Leibniz's mechanics. These quantities introduced explicitly by Newton appeared in Leibniz's theory as *hidden variables* or parameters. Having rejected the idea of inertia, Leibniz was forced to replace the notion of *inert mass* by forces whose most important representative had been called "living forces" and determinate by the product of mass and the square of velocity $m \cdot v \cdot v$ [Leibniz, Brevis], [Leibniz, Specimen]. Then excluding the possibility of perpetual motion, Leibniz postulated that there "is neither more nor less portency in the effect than in the cause" [Leibniz, Specimen, I (11)],[60] [Leibniz, Specimen, UV (11)][61] and studied the redistribution of living forces among the interacting bodies. The postulate is analytically formulated as an equation comprising and connecting the living forces *before* and *after* the impact (interaction). The time of interaction is not taken into account, but indirectly included in the difference between the initial and final values of the velocity. The magnitude of difference should be related to the duration and the intensity of interaction. The intensity of the interaction is represented by the *force* and is independent of the mass of the bodies whereas the *duration* of interaction is either of infinitesimal or of finite magnitude. Hence, a finite change of velocity is necessarily generated in a finite time interval (of interaction) whereas an infinitesimal change of velocity (being either an increment or decrement) is as necessarily generated in an infinitesimal time interval. Postulating the equality

$$m_1 v_{1\,\text{before}}^2 + m_2 v_{2\,\text{before}}^2 = m_1 w_{1\,\text{after}}^2 + m_2 w_{2\,\text{after}}^2 = \text{const}, \tag{2.1}$$

Leibniz had indirectly assumed that there was a beginning and an end of interaction whose magnitudes, following Newton, cannot be represented by finite increments, but only by the ratio of infinitesimal increments or first and last ratios of the "errors" and the square of augment of time, i.e. the ratio dds/dt^2. Hence, also Newton made use of infinitesimal quantities *ds, dds* and *dt* in advance, i.e. before their ratio had been finally taken into account.

Obviously, the increments/decrements of velocities or the differences "before interaction" and "after interaction" $|v_{\text{before}} - w_{\text{after}}| = \Delta v$ played the role of "hidden parameters". Moreover, although Leibniz introduced derivative forces being generated

[60] "(...) nec plus minusve potentiae in effectu quam in causa contineatur." [Leibniz, Specimen, I (11)]

[61] "(...) sed effectum plenum esse causae integrae aequalem." [Leibniz, Specimen, UV (11)]

by the impact of bodies, these forces are not explicitly represented. However, making use of the calculus of differences the relation is obtained $\Delta v_1 \sim K_{12} \cdot \Delta t_{\text{impact}}$ and it follows: (i) the forces are generated by interacting bodies, (ii) hence K_{12}, K_{21} are to be indicated by both the bodies involved in the interaction, (iii) the change of living forces is given by $\Delta(v \cdot v) = 2 \cdot v \cdot \Delta v$, (iv) hence, the change of living forces $v_1 \Delta v_1 \sim K_{12}$ can also be represented by the previously introduced forces. Hence, Newton and Leibniz invented different representation of one and the same quantity, the change in velocity Δv which is generated due to the interaction of bodies. These results were only analytically completely demonstrated by Euler [Euler E015/016, §§ 125–152].

2.3.1 Early Version

Leibniz's program for mechanics is embedded into a general program for a general science (scientia generalis). From the very beginning, mechanics is not only explicitly related to geometry, arithmetic, logic and metaphysics, but methodologically based on these disciplines.[62] The fundamental methodological relation is that physics had been assumed to be *subordinate* to mathematics, i.e. arithmetics or algebra and geometry, and metaphysics. The non-physical basic notions for physics are (i) magnitude, (ii) position (situs)[63] and (iii) resistance or action and suffering.

> Physica est Arithmetica sive Algebrae subordinata quatenus agit de magnitudine; Geometricae quatenus agit de situ; Metaphysica quatenus de resistentia sive actione et passione. [Leibniz, A VI, 267][64] (1679)

The foundation of mechanics by merging arithmetics, geometry and metaphysics had been preserved in the following decades. In the *Discours de métaphysique* Leibniz presented mechanics as a result of a unification of principles in terms of *fundamental* and *subordinate* rules [Leibniz, Discours],[65] later interpreted in terms of necessary and contingent truths [Leibniz, Monadology]. Neither time nor motion can ever exist as a whole,

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[62] Although Leibniz invented the mathematical algorithm for differentials and differentio-differentials and higher order differentials, the definition of velocity by the ratio of differentials of space and time, $ds = v\,dt$ or $ds/dt = v$, had been not discussed by Leibniz, but only by Varignon in 1700 [Varignon 1700].

[63] Later in 1715, Leibniz defined "situs" as a "modus of coexistens" made up of magnitude and relative position. "Situs est coexistentia modus. Itaque non tantum quantitatem, sed et qualitatem involvit." [Leibniz, Initia, p. 354]

[64] "Physics is subordinated to Arithmetics or Algebra as far the magnitude is concerned and to Geometry as far as the position [defined topologically and metrically] is concerned further to Metaphysics as far as the resistance or action and suffering are concernd." [Leibniz, A VI, 267] (1679)

[65] "17. An example of the subordinate rule of natural law, which shows that God always systematically conserves the same *force*, but not (contrary to the Cartesians and others) the same quantity of *motion*." [Leibniz, Discours] Here, Leibniz stressed the difference between "motion" and "force". "17. Exemple d'une Maxime subalterne du loy de Nature, où il est monstré que Dieu conserve tousjours regulierement la même force, mais non pas la même quantité de mouvement, contre les Cartesiens et plusieurs autres." [Leibniz, Discours]

> (...) because a whole does not exist if it has no coexisting parts. Thus there is nothing real in motion itself except that momentaneous which must consist of a force striving toward change. From that originates also all this being in corporeal nature except the subject of geometry, i.e. the extension. By this argumentation, the truth and the doctrines of the ancients are simultaneously taken into account. [Leibniz, Specimen, I (1)][66]

Leibniz compared the saving of the ideas of Plato and the Stoics in the age of Democritus to the saving of the doctrines of peripatetics on the forms and entelechies in present times [Leibniz, Specimen, I (1)].[67]The main discrepancy between Newton's and Leibniz's mechanics is due to the different assumptions on the conservation of mechanical quantities. Following Descartes who considered the "quantity of motion" as being conserved in the world, Leibniz alternatively constructed the conservation of another quantity called "living forces" [Leibniz, Brevis].[68] This principal dissimilarity was well known and caused a long lasting debate on the true measure of forces originating from Leibniz's *Brevis memorabilis* [Leibniz, Brevis] published in 1686 one year before Newton's *Principia*. Leibniz' main intention was to demonstrate that perpetual motion is to be excluded [Leibniz, Specimen, I (11)]. In 1738, Voltaire commented:

> Descartes, sans faire mention de la force, avançait sans preuve qu'il y a toujours quantité égale de mouvement; et son opinion était d'autant moins fondée que les lois mêmes du mouvement lui étaient absolument inconnues.
>
> Leibnitz, venu dans un temps plus éclairé, a été obligé d'avouer, avec Newton, qu'il se perd du mouvement; mais il prétend que, quoique la même quantité de mouvement ne subsiste pas, la force subsiste toujours la même.
>
> Newton, au contraire, était persuadé qu'il implique contradiction que le mouvement ne soit pas proportionnel à la force. [Voltaire, Élémens, Chap. IX]

In 1686, Leibniz emphasized the relation between "forces" or "metaphysical principles" and the "phenomena of bodies" or the "phenomena in nature" described by geometry or the "science of extension". Leibniz claimed that geometry is necessary, but not sufficient to describe the interaction of bodies since except extension. There should be additional principles of action and suffering. These additional principles are represented by a variety of different forces among them those of purely metaphysical origin like the *primitive* forces [Leibniz, Specimen] and claimed that the

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66 "Nihilique adeo in ipso reale est, quam momentaneum illud, quo in vi ad mutationem nitente constitui debet. Huc igitur redit quicquid in natura corporea praeter Geometriae objectum seu extensionem. Eaque demum ratione simul et veritati et doctrinae veterum consulitur." [Leibniz, Specimen, I (1)]

67 Geometry is the science of extension. Following Newton who recovered the ancient prototype for the relation between points, lines, surfaces and solids (compare Chap. 3), extension (extended things) is generated by motion, hence, mechanics is the science how extension is generated by motion whereas, following Leibniz, mechanics is the science how extension is generated by forces [Leibniz, Specimen, I (1)]. Later in 1715, Leibniz adopted also the Newtonian model [Leibniz, Initia].

68 "18. La distinction de la force et de la quantité de mouvement est importante, entre autres pour juger qu'il faut recourir à des considérations métaphysiques séparées de l'étendue à fin d'expliquer les phénomènes des corps." [Leibniz, Discours]

"phenomena would be completely others" if "mechanical rules would only depend on geometry".[69]

Following Leibniz and applying the model of the infinity of possible worlds being different from each other [Leibniz, Monadology, §§ 53–58], the phenomena depend on the world where they appear, whereas, because of its necessity, the *extension* or the *geometry* should be the same in different worlds. The extension is an invariant property of all possible worlds. Hence, the extension is also a necessary property of the existing world.[70] The phenomena are generated by the motion of corporeal things. The derivative forces are related to those corporeal motions which result in an interaction between bodies [Leibniz, Specimen, I (4)].[71] One may ask whether there is one possible world where the phenomena are *independent* of metaphysics, but depend only on geometry. The answer may be obtained from Leibniz's model how the different places in the continuum or plenum may be distinguished from each other.[72]

In all later writings, Leibniz extended the frame of basic concepts which had been established very early between 1675 an 1679.[73] These attempts had been continued in the next decade until 1686. The program for mechanics is based on the investigation of the problems to be solved with respect to (i) composition of the continuum, (ii) time, (iii) place, (iv) motion, (v) atoms, (vi) indivisibles and (vii) infinity. Leibniz did not modify this program as far as the relation between geometry, arithmetic and metaphysics is concerned, but modified essentially the assumptions on the basic properties of bodies, i.e. mainly the concept of inertia [Leibniz, Specimen, I (10)].

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[69] . "21. Si les réglés mécaniques dépendoient de la seule Géométrie sans la métaphysique, les phénomènes seroient tout autres. " [Leibniz, Discourse]

[70] His assumption is not in contradiction with the model that the "forces is prior to extension" which is not related to the plenum, but to the corporeal things in the world, "in rebus corporeis". "In rebus corporeis esse aliquid praeter extensionem, imo extensione prius, alibi admonuimus, nempe ipsam vim naturae ubique ab Autore inditam." [Leibniz, *Specimen*, I (1)]

[71] "Vim ergo derivativam, qua scilicet corpora actu in se invicem agunt, aut a se invicem patiuntur, hoc loco non aliam intelligimus, quam quae motui (locali scilicet) cohaeret, et vicissim ad motum localem porro producendum tendit. Nam per motum localem caetera phenomena materialia explicari posse agnoscimus. Motus est continua loci mutatio, itaque tempore indiget." [Leibniz, *Specimen*, I (4)]. Relative motion is not something real or absolute, "quasi reale quiddam esset motus et absolutum", hence, the phenomena are described with respect to a frame of reference which is not "given", but "chosen" (therefore "contingent"). "Sic igitur habendum est, si corpora quotcunque sint in motu, ex phaenomenis non posse colligi in quo eorum sit motus absolutus determinatus vel quies, sed cuilibet ex iis assumpto posse attribui quietem ut tamen eadem phaenomena prodeant" [Leibniz, Specimen, II (2)].

[72] This "world" consists only of the plenum ("le plein étant supposé") or the "extended thing" without borders [Leibniz, Monadology, § 8]. The vacuum is automatically excluded. There is no empty space [Leibniz, Monadology, § 69]. Moreover, the plenum is completely free of monads. As a consequence, there are no different parts which can be distinguished from each other, i.e. the "world does not consist of coexisting parts". This property is also assigned to "motion" and "time" [Leibniz, Specimen, I (1)].

[73] Compare the analysis of Leibniz's interpretation of differentials by Arthur [Arthur, Syncategorematic].

In contrast to later version, extension appeared in the early theory of motion in two modifications, first as the extension of a body and second as the extension of vacuum.

> Corpus est extensum resistens. Extensum est quod habet magnitudinem et situm. Resistens est quod agit in id a quo patitur. Vacuum est extensum sine resistentia. [Leibniz, A VI, 267][74]

In the further development, the "resisting thing which acts in those others from whose it suffers" is specified in terms of "active and passive forces" [Leibniz, Specimen]. Unfortunately, the change in the terminology results in the decomposition of the previously correlated and complementary parts of the "action" which always simultaneously appear in the impact of bodies. The alternative possible interpretations are the following: (i) The bodies are *permanently acting* and *suffering* and (ii) the bodies are not permanently acting and suffering or not permanently resisting, but only in occasion of the impact or interaction. The theorem (i) had been developed by Leibniz in the *Specimen*. Guided by the idea, that the science of the ancients is the science of equilibrium or statics where permanently acting forces impressed upon bodies are compared to each other by joining the bodies in the lever, the science of motion should be necessarily also a science of forces. Following Leibniz: (i) body is the extended resisting thing, resisting is what acts upon that from which it suffers and (ii) a vacuum is an extended thing without resistance. The uniform motion is out of the scope of a theory where the velocity of a body is correlated with forces. The directions of forces are indeterminate. Newton assumed that the change in motion is determinate by the direction of the "impressed moving force". Following Euler, the directions of the forces in case of interaction are determinate with respect to the plane whose orientation geometrically represents the necessary conditions for the interaction of two bodies [Euler E842, §§ 69 and 71].

2.3.2 Later Version: Living Forces

In 1686, Leibniz published a criticism of Descartes' postulate on the equivalence of "moving force" and "quantitiy of motion" ("qui *vim motricem* et *quantitatem motus* pro re aequivalente habebat" [Leibniz, Brevis]). In 1686 and 1687, Leibniz published a short note on the errors of Descartes [Leibniz, Brevis][75] whereas Newton

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[74] "A body is an extended resisting thing. An extended thing is that magnitude and place has. Resisting is that acts in those others from whose it suffers. Vacuum is an extended thing without having resistance." [Leibniz, A VI, 267] (1679–1681)

[75] The famous *relational theory of time and space* had been systematically developed by Leibniz only later in the treatise *Initia rerum mathematicarum metaphysia* in 1715 [Leibniz, Initia] and in the correspondence with Clark between 1715 and 1716 [Leibniz, Clarke]. In the 1715 paper on the foundation of mathematics, Leibniz did not consider bodies and forces as mechanical objects, but bodies only as geometrical, i.e. extended things. One of the main subjects is the relation between point, line, surfaces and solids (compare Chap. 3). Motion is treated as a *translation* of geometrical objects like *points* and *lines* which is performed with an *indeterminate* velocity.

published a comprehensive treatise on the *Mathematical principles of natural philosophy* which was also based on a carefully performed analysis and criticism of Descartes' principles, respectively. The main subject of both treatises is the relation between the motion of bodies and forces of nature where motion is considered as *phenomena* whereas forces are different from phenomena where the latter have to be investigated by studying the phenomena [Newton, Principia].

In both writings it was assumed that bodies move (i) either with constant velocity in one and the same direction which was already postulated by Descartes [Descartes, Principles] (ii) or, if the bodies perturb each other, they refuse to continue their previous motion due to the presence and action of "impressed moving forces", as Newton explained, or, as Leibniz explained,[76] due the forces which enable "mutual actions" of bodies during the impact,[77] or, due to the generation of forces by interacting bodies, as Euler explained later [Euler E842]. In 1686, Leibniz called the measure of the force of bodies "moving force" ("vis motrix") to distinguish this concept from the Cartesian "quantity of motion" ("quantitas motus"). Later in 1695, Leibniz renamed this force by "living force" ("vis viva") which may be generated by removal of an obstacle and whose appearance was observed in the up and down motion of a weight[78] [Leibniz, Specimen]. Following Leibniz, the idea of a living force is related to the "dead force" ("vis mortua") which had been studied in the theory of equilibrium by the ancients.[79] The "quantity of motion" in the whole nature can neither be increased nor decreased.

> Itaque cum rationi consentaneum sit, eandem motricis potentiae summam in natura conservari, et neque imminui, (...) neque augeri, quia vel ideo motus perpertuus mechanicus nuspiam succedit, (...). [Leibniz, Brevis] (A 1686)

The Leibnizian concept had been later developed by Châtelet [Châtelet, Institutions] and d'Alembert [d'Alembert, Traité] who explained the living forces by the ability of the moving body to remove partially or completely obstacles the body met whereas the Cartesian measure is appropriate for measuring the pressure which is due to an invincible obstacle. This distinction had been introduced by Leibniz who called the second type of forces "dead forces" to stress the difference to "living forces" [Leibniz, Specimen, I (6)].

> Hinc *Vis* quoque duplex; alia elementaris, quam et *mortuam* appello, quia in ea nondum existit motus, sed tantum solicitatio ad motum, (...) alia vero vis ordinaria est, cum motu actuali conjuncta, quam voco *vivam*. [Leibniz, Specimen, I (6)] (A 1695)

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[76] Leibniz based the theory on the *differences* between bodies, i.e. their different velocities. Hence, the *change* of motion is not regarded as independent of the velocities, but directly related to the *nisus*.

[77] Leibniz called these forces "derivative forces". "Vim ergo derivativam, qua scilicet corpora actu in se invicem agunt, aut a se invicem patiuntur, hoc loco non aliam intelligimus, quam quae motui (locali scilicet) cohaeret, et vicissim ad motum localem porro producendum tendit." [Leibniz, Specimen, I (4)]

[78] This construction was stimulated by Huygens' pendulum. Leibniz removed the horizontal motion and considered only the pure up and down motion of the weight.

[79] "Veteres, quantum constat, solius vis mortuae scientiam habuerunt, eaque est, quae vulgo dicitur Mechanica, agens de vecte, trochlea, plano inclinato." [Leibniz, Specimen, I (8)]

Comparing (A1686) to (A1695) a remarkable difference should be highlighted since in the earlier version the body is assumed to move in both cases whereas in the later version Leibniz compared a body which is only *prepared for motion*, but is not moving to a body which is really *moving*. In 1695, though the assumed forces cannot be represented in the intuition, Leibniz based the theory on the notion of an inherent force. In 1698, Leibniz commented on this topic in the writing *De ipsa natura sive de vi insita actionibusque creaturarum* [Leibniz, De ipsa].[80]

> Haec autem vis insita distincte quidem intellegi potest, sed non explicari imaginabiliter. [Leibniz, De ipsa, § 7]

The emergence of motion is regarded as criterion for the existence of such force when motion sets in if an external obstacle is removed which previously hindered the body to move. In the 1698 paper *De Ipsa* which may be read as a reaction to Newton's *Principia*, Leibniz reconsidered the concept of forces.

> (...) ita verbum benedictionis non minus mirificum aliquam post se in rebus reliquisse producendi actus suos operandique foecunditatem nisumve, ex quo operatio, si nihil obstet, consequatur. [Leibniz, Ipsa, § 8]

Following Kepler and substituting Newton's *vis inertia* with Kepler's concept of *inertia*, Leibniz assumed a natural inertia as an inherent force which made the body able to rest (vis passiva resistendi [Leibniz, De ipsa, § 11]) which had been later also assumed by Châtelet [Châtelet, Institutions]. Methodologically, a correlation should be established between (i) active (moving) – passive (resisting) and (ii) primitive and derivative forces.

Very early Leibniz rejected the idea that "the nature of bodies is solely determined by the inert mass" [Leibniz, Specimen, I (10)]. Instead of the inert mass Leibniz introduced a concept of *complementary* forces, mainly the "active and passive forces". Both the forces are further specified by the same subdivision into "primitive active" and "derivative actives" as well as "primitive passive" and "derivative passive" forces. The distinction is finally based on another distinction[81] which is due to "*internal* and *external*". The "primitive active" force is an inherent force whereas the "derivative active" forces results from the limitation of the first one appearing in the impact of bodies [Leibniz, Specimen, I (3)].

> Duplex autem est *vis Activa* (...) nempe aut *primitiva*, quae in omni substantia corporeaper se inest (...) aut *derivativa*, quae primitivae velut limitatione per corporum inter se conflictus resultans, varie exercetur. [Leibniz, Specimen, I, (3)]

[80] For the development of Leibniz's concept of forces before 1695 compare Duchesneau [Duchesneau].

[81] Behind this distinction is the principle of sufficient reason based on the assumption of contingent truths [Leibniz, Monadology, §§ 31–35]. "Internal" and "external" are contingent relations of finite things. In contrast, rest and motion are not considered as basically contingent properties since "cum corpus omnimode quiescens a rerum natura abhorrere arbiter" [Leibniz, Specimen, I (3)]. Châtelet also assumed a similar difference between rest and motion [Châtelet, Institutions]. Therefore, following Descartes in assuming extension as basic property, Châtelet modified Descartes' assumption on rest and motion in favour of Leibniz.

Leibniz completed the Cartesian approach based solely on geometry by the introduction of forces. However, none of these forces can be considered as "necessary" since their generation is guided by contingency. The only candidate for being as necessary as the extension was the inertia which had been rejected by Leibniz. Being aware of this gap Leibniz replaced the previous concept of inertia by the conservation of "living forces" where the shadow of inertia is entering as the numerical value of the masses of bodies involved in the impact,[82] however, an explicit expression for the *change in motion* where the "derivative forces" should appear is missing. Hence, Leibniz represented the change in motion in terms of finite differences between the initial and the final velocities being assigned to the body before and after the impact, respectively. (Compare Eq. 2.1). The mass is represented by a numerical value [Couturat, Cassirer],[83] but it is not related to any of the forces Leibniz discussed. A direct relation between mass and forces is missing in Newton's 2nd Law (compare Corollaries 1 to 5 in Sect. 2.2.3). Newton assumed that the "inherent force of inertia" is activated by the "impressed moving force" [Newton, Principia, Definitions]. The quantitative definition of the mass is due to density times volume.[84]

In 1686, Leibniz distinguished between the Cartesian *quantity of motion* (quantitas motus) and the *force motion* (vis motrix) [Leibniz, Brevis]. It is quite clear whether the Cartesian concept is considered as invalid or as valid, it has another range of validity. Leibniz based his argumentation on the same principle as Descartes did.

> Itaque cum rationi consentaneum sit, eandem motricis potentiae summam in natura conservari, et neque imminui, quoniam videmus nullam vim ab uno corpore amitti, quin in aliud transferatur, neque augeri, quia vel ideo motus perpetuus mechanicus nuspiam succedit, quod nulla machina ac proinde ne integer quidem mundus suam vim intendere potest sine novo externo impulsu; inde factum est, ut Cartesius, qui *vim motricem* et *quantitatem motus* pro re aequivalente habebat, pronunciaverit eandem quantitatem motus a Deo in mundo conservari. [Leibniz, Brevis]

Here, Leibniz stressed the conformity with Descartes as far as the conservation of certain quantities is concerned. The difference to Newton's treatment of mechanics published only one year later in 1687 is already automatically included

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[82] The impact is always considered as the mutual action of bodies [Leibniz, Specimen, I (10)]. "Atque hoc est quod experimur, eundem nos dolorem sensuros sive in lapidem quiescentem ex filo si placet suspensum incurrat manus nostra, sive eadem celeritate in manum quiescentem incurrat lapis." [Leibniz, Specimen, II (2)] Here, Leibniz assumed the universality of relative motion derived from the impact of bodies. The only missing conclusion is that the impact is only possible due to the *inertia* of both interacting bodies.

[83] "Pour définir la force vive, c'est-à-dire précisément la quantité qui se conserver dans le choc élastique et qui déterminer par suite la marche ultérieure des mobiles, il fallait tenir compté du facteur *masse*, sans lequel on ne peut l'équivalence des forces vives échangées par le choc. L'invention du concept de masse ne constituait pas seulement un progrès capital de la mécanique: elle permettait à Leibniz de dissocier complètement l'idée de matière de l'idée d'étendue, puisque le coefficient appelé masse est une quantité numérique, et non une grandeur spatiale." [Couturat, Cassirer]

[84] "*The quantity of matter is the measure of the same, arising from its density and bulk conjunctly.* Thus air of double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity." [Newton, Principia, Definitions]

since Newton claimed that neither of the mechanical quantities are conserved [Newton, Principia]. Therefore, Leibniz introduced the notions of (i) "force" and "moving force" before Newton invented the terminus (ii) "impressed moving force". The difference is striking since (i) it is assigned to a body moving with a certain velocity whereas (ii) it is explicitly related to the "change of velocity". In Leibniz's model based on Huygens' experiments the *change* of motion is treated as an *intermediate* step or time interval which connects (a) two states S1 and S2 of one and the same body A being essentially different from each other, an initial state and a final state, respectively, and, additionally, the states of two bodies, A and B, having different masses. Descartes' measure is also related to two bodies C and D set in motion and being in different states.

In 1695, Leibniz introduced a variety of different forces which are finally, in 1698, reduced to two basic types, internal and external forces. The external forces are represented by *obstacles* hindering motion whereas the internal forces are defined by the permanent tendency to *act*, subsequently but not primarily, divided into the ability to *act* and to *suffer*.

> (...) ita verbum *benedictionis* non minus *mirificum* aliquam post se in rebus reliquisse producendi actus suos operandique foecunditatem nisumve, ex quo operatio, si nihil obstet, consequatur. Quibus addi potest, quod alibi a me explicatum est, etsi nondum fortasse satis perspectum omnibus, ipsam rerum substantiam in agendi patiendique vi consistere: unde consequens est, ne res quidem durabiles produci posse, si nulla ipsis vis aliquamdiu permanens divina virtute imprimi potest. [Leibniz, De ipsa, § 8]

The latter statement is in complete opposition to Newton who introduced the inherent force, called "vis insita", being the cause of the *preservation* of state, i.e. representing the *absence* of any action or suffering. Moreover, this force is not permantenly present, but has to be activated by the force impressed upon the body. Nevertheless, despite the differences there is a common basis in Newton and Leibniz which is mainly caused by the common origin of the problems in matematics and mechanics. Although from the outer view Newton's and Leibniz's mechanics looked differently, the common indispensable basis of the theory remains to be the geometry.

This common origin had been preferentially noticed by those predecessors who belong to the young generation of scholars in the first half of the 18th century. Later, following in goal and spirit Newton and Leibniz and demonstrating the common basis instaed of the non-compatible parts of their theories, Châtelet is not ready to replace *completely* geometry with the calculus.[85] Here, Châtelet is following Descartes, but she does not follow the Cartesians who reduced all properties of

[85] Here, we find a controversy which can be also observed in 20th century physics. In contrast to Euler, the development of quantum mechanics had not been guided directly by mathematics. However, assisted by Klein, Schrödinger recalled into the memory the geometrical representation of motion by Hamilton. Even in 1891, Klein tried to stimulate physics community to make use of Hamilton's theory. The response was disappointing. Only Schrödinger was ready to accept the invitation and, not surprisingly, was successful in developing independently of Heisenberg a new theory, called quantum mechanics. "Eines genetischen Zusammenhangs mit Heisenberg bin ich mir durchaus nicht bewußt. Ich hatte von seiner Theorie natürlich Kenntnis, fühlte mich aber durch die mir sehr schwierig scheinenden Methoden der transzendenten

the body to extension. The consequence is that the *extension* is considered as basic property of *all bodies* and the difference between bodies and other extended things[86] has to be introduced by the assumption of forces [Châtelet, Institutions, §§ 138–147]. This is the procedure of all followers of Descartes. Newton intoduced the forces of inertia and impressed moving forces [Newton, Principia]. Leibniz introduced active and passive, primitive and derivative, living and dead, total and partial and respective and directive (common) forces [Leibniz, Specimen].

> 145. Tous les changements qui arrivent dans les Corps peuvent s'expliquer par ces trois principes, *l'étendue, la force résistante*, & *la force active*; (...). [Châtelet, Institutions, § 145]

The followers of Newton and Leibniz had different, but correlated approaches at their disposal:

A. Newton's body-space (absolute space, absolute rest and motion) approach,
B. Newton's body-body approach (relative rest and motion),
C. Newton's body-force approach,
D. Newton's force-force model, the inherent force is excited by the impressed moving force,
E. Leibniz's relational body-body approach (obtained from (A)),
F. Leibniz's original force-force approach from 1695 (also related to (A)),
G. The modification of primitive moving forces inherent the bodies by the interaction of bodies and
H. The combined body-body and force-force approach from 1698.

Euler based his mechanics on (A) and (B) rejecting Newton's force of inertia and preserving only Leibniz's derivative forces as the only kind of forces investigated in mechanics. From the assumed body-body relations it follows that all the forces are only generated due the interaction of bodies. The theory of relative motion and the theory of the changes in motion had to be developed in parallel. Euler developed a complete relativistic theory within the frame of classical mechanics (compare Chap. 6).

Châtelet based the *Institutions* on (A) as far as extension is concerned and on (C) and (D) as far as the relation between active and passive forces are concerned. Therefore, Châtelet concentrates on the relation between dead and living forces. The theory of motion and rest is treated separately where the relational part is dominating in comparison to the relativistic elements. The most important force is the inherent "primitive active force" [Leibniz, Specimen, I (2)].

Algebra und durch den Mangel an Anschaulichkeit abgeschreckt, um nicht zu sagen abgestoßen" [Schrödinger, Heisenberg]. However, the mathematician David Hilbert was aware of the common features of Heisenberg and Schrödinger's approaches in advance [Thall's history of quantum mechanisc. http://mooni.fccj.org/∼ethall/quantum/quant.htm]. However, Hilbert's message was not listened by Heisenberg.

[86] Euler distinguished between *space* and *bodies* as *extended* things claiming that this difference cannot be explained by forces since rest and uniform motion of bodies are not related to forces [Euler E842 Chap. 2] (compare Chap. 4).

Chapter 3
Newton and Leibniz on the Foundation of the Calculus

Tum vero nullum est dubium, quin NEUTONO eam calculi differentialis partem, quae circa functiones irrationalis versatur, acceptem referre debeamus; (...) *LEIBNIZIO autem non minus sumus obstricti, quod hunc calculum, antehac tantum velut singulare artificium spectatum, in formam discipline redegerit, eiusque praecepta tanquam in systema collegerit, ac dilucide explicaverit.*

Euler E212, Preface LXIII[1]

By the help of the new Analysis Mr. Newton found out most of the Propositions in his Principia Philosophiae: but because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically, he demonstrated the Propositions synthetically, that the system of the Heavens might be founded upon good Geometry. And this makes it now difficult for unskilled Men to see the Analysis by which those Propositions were found out.

Newton, Account, p. 20

The calculus had been represented by Newton and Leibniz in different versions.[2] These representations may be also considered as different foundations of the calculus. Newton preferred a geometrical foundation which is based on the continuous generation of lines, surface and solids [Newton, Quadrature (Harris)]. Leibniz

[1] There is no doubt that Newton has invented that part of the calculus which refers to irrational functions. However, we are no less indebted to Leibniz who transformed the calculus which was before him a kind of art into the form of a science. "(...) kannte man das letzte Verhältniß ihrer verschwindenden Incremente schon lange vor *Newton* und *Leibnitz*. (...) Dagegen leidet es keinen Zweifel, daß wir denjenigen Theil des Differenzial=Calculus, welcher sich mit den irrationalen Funktionen beschäftigt, *Newton* verdanken, (...) *Leibnitzen* sind wir nicht weniger verpflichtet. Er brachte nemlich diesen Calcul, den man bis dahin bloß als einen besonderen Kunstgriff betrachtet hatte, in die Form einer Wissenschaft, bildete aus den Regeln desselben ein System, und stellte dasselbe in einem hellen Lichte dar." [Euler E212, LXXII (Michelsen)]

[2] The models are to be distinguished from the non-standard analysis developed by Robinson [Robinson], Laugwitz and Schmieden [Schmieden Laugwitz], [Laugwitz, Nonstandard], [Laugwitz, Zahlen] in the 20th century. The 19th century standard model had been founded by Weierstraß mainly based on the investigations of d'Alembert and Cauchy. These authors treated the calculus as a purely mathematical discipline without any reference to its physical implications. However, following Leibniz's interpretation [Leibniz, Dangicourt] the metaphysical implication was accentuated by Cantor [Cantor].

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invented a geometry related approach based on the analysis of *curved* and *straight* lines (tangents) [Leibniz, Nova methodus] and, additionally, the method of differences and sums which is not immediately related to geometry, but is founded on the consideration of *series of numbers* whose *differences* and *sums* are the basic quantities [Leibniz, Elementa], [Leibniz (Child)].[3] Considering the calculus as a set of algorithms for operations with mathematically defined quantities there are different modes of interpretation by auxiliary disciplines being different from mathematics. The main auxiliary non-mathematical discipline is *mechanics*. In mechanics as well as in mathematics, *infinitesimal* and *finite* quantities had to be distinguished, but denoted by other names, such as (i) Newton's model of fluents, fluxions and moments related to time, (ii) instant and time, (iii) point and space, (iv) sollicitations (nisus) and impetus, (v) beginning of motion[4] and the end motion[5] and (vi) the relation between dead and living forces. All these concepts had been discussed by Leibniz [Leibniz, Specimen] (compare Chap. 2).

According to the discipline whose principles had been used to deliver a criterion for confirmation of the applicability of algorithms by theses models one can distinguish foundations[6] in terms of (i) geometrical, (ii) mechanical subdivided into two classes, (a) time, space and motion, (b) masses and forces introduced by Leibniz [Leibniz, Specimen, II (2)], (iii) arithmetical and (iv) metaphysical origin. This difference between the *algorithm* and its *interpretation* becomes a characteristic feature of the development of physics[7] and had been continued until present where new

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[3] "Differences and sums are the inverse of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series; and I enunciate the former thus, $\int dx = x$, and the latter thus, $\int dx = x$." [Leibniz, (Child), p. 142]

[4] "Cor. 3. The same thing is to be understood of any space whatsoever described by bodies urged with different forces. All which, in the very beginning of motion, are as the forces and the squares of the times conjunctly." [Newton, Principia] (compare Chap. 2)

[5] Rest is an evanescent velocity [Leibniz, Specimen].

[6] One of the earliest reactions to Leibniz's *Nova methodus* is due to Newton who commented in 1687: "In literis quae mihi cum Geometra peritissimo G. G. Leibnitio annis abhinc decem intercedebant, cum significarem me compotem esse methodi determinandi Maximas & Minimas, ducendi Tangentes, & similia peragendi, quae in terminis surdis aeque ac in rationalibus procederet, & literis transpositis hanc sententiam involventibus [Data aequatione quotcunque; fluentes quantitates involvente, fluxiones invenire; & vice versa] eandem celarem: rescripsit Vir Clarissimus se quoque in ejusmodi methodum incidisse, & methodum suam communicavit a mea vix abludentem praeter quam in verborum & notarum formulis. Utriusque fundamentum continetur in hoc Lemmata." [Newton, Principia, Book II, Sect. II, Lemma II, Scholion, pp. 253–254] Hence, "the method or the analytical representation is the same, but semantics is different". In 1687, there is no reference to Cavalieri. Later in 1714, Newton accentuated the *different origin* and the *different interpretation* and stated that Leibniz made use of Cavalieri's indivisibles which are excluded by the method of fluxions [Newton (Collins), Commercium].

[7] The development of the theory and its subsequent application and interpretation may be compared to the development of quantum mechanics in the 20th century (compare Chap. 8). Surprisingly, the general features are very similar since in both cases there are little doubts in the reliability of *algorithms* and the confirmation of the results either mathematically or experimentally, but the interpretation of the basic principles is still controversially. Before 1925, there are different interpretations by (I) Planck, (II) Einstein, (III) Bohr and Sommerfeld, (IV) Schrödinger 1923, after 1925 and 1926 there are the interpretations by (a) Hilbert, (b) Bohr and Heisenberg,

developments appeared in the last decades. The origin of some of them can be traced back to the early criticism of the calculus by Nieuwentijt [Nieuwentijt, Analysis] in 1695.[8] In 1666, Newton introduced fluents and fluxions based on a "universal infinitesimal quantity" having the dimension of time [Newton, Method of Fluxions]. In 1714, Newton accentuated the fundamental role of geometry for the understanding of the construction of the world and the foundation of the method of fluents and fluxions:

> And whereas it has been represented that the use of the letter *o* is vulgar, and destroys the Advantages of the Differential Method: on the contrary, the Method of Fluxions, as used by Mr. Newton, has all the Advantages of the Differential, and some others. It is more elegant, because in his Calculus there is but one infinitely little Quantity represented by a Symbol, the Symbol *o*. We have no Ideas of infinitely little Quantities, and therefore Mr. Newton introduced Fluxions into his Method, that it might proceed by finite Quantities as much as possible. It is more Natural and Geometrical, because founded upon the *primae quantitatum nascentium rationes*, which have a Being in Geometry, whilst *Indivisibiles*, upon which the Differential Method is founded, have no Being either in Geometry or in Nature. [Newton, Account]

Obviously, time is intimately related to the continuous flux and plays a dominant role. By this assumption which is the counterpart of Leibniz's principle of continuity, Newton excluded any interruptions or jumps and ensured that the quantities are not composed of parts like Cavalieri's indivisibles and Leibniz's differentials.[9]

Newton developed the basic principles of the calculus at the beginning of his scientific career between 1665 and 1666, but published some parts of his papers only with a considerable time delay of 40 years in the appendix to the treatise *Opticks* in 1704. In 1687, Newton published the famous *Mathematical Principles of Natural Philosophy* which was based (i) on the investigations previously summarized in *De motu* [Newton, De motu] and (ii) on the mathematical methods of fluents

(c) Schrödinger, (d) Born and (e) the criticism by Einstein. The main directions after 1927 are the following: "(i) Ensemble interpretation, or statistical interpretation, (ii) The Copenhagen interpretation, (iii) Consciousness causes collapse, (iv) Consistent histories, (v) Objective collapse theories, (vi) Many worlds, (vii) The decoherence approach, (viii) Many minds, (ix) Quantum logic, (x) The Bohm interpretation, (xi) Transactional interpretation, (xii) Relational quantum mechanics, (xiii) Modal interpretations of quantum theory, (xiv) Incomplete measurements." [http://en.wikipedia.org/wiki/Interpretation_of_quantum_mechanics]

[8] Currently, the following mathematical and logical interpretation had been discussed: "(i) Differentials as linear maps. This approach underlies the definition of the derivative and the exterior derivative in differential geometry. (ii) Differentials as nilpotent elements of commutative rings. This approach is popular in algebraic geometry. (iii) Differentials in smooth models of set theory. This approach is known as synthetic differential geometry or smooth infinitesimal analysis and is closely related to the algebraic geometric approach, except that ideas from topos theory are used to hide the mechanisms by which nilpotent infinitesimals are introduced. (iv) Differentials as infinitesimals in hyperreal number systems, which are extensions of the real numbers which contain invertible infinitesimals and infinitely large numbers. This is the approach of nonstandard analysis." [http://en.wikipedia.org/wiki/Differential_(infinitesimal)]

[9] However, in the later criticism by Berkeley, Newton's sophisticated distinctions had been ignored and both versions were treated as equivalent, i.e. as incorrectly and insufficiently demonstrated and full of contradictions [Berkeley, Analyst].

and fluxions. However, the scientific community did not get knowledge of the new method from Newton's, but from Leibniz's paper *Nova methodus* published in 1684 [Leibniz, Nova methodus]. A short paper by Newton *De analysi per aequationes infinitas* (On the analysis of infinite series) [Newton, Analysi] appeared in 1669. The treatise *De methodis serium et fluxium* (On the method of series and fluxions) [Newton, Methodis] was written 1671, but only published 1736 and the paper *De quadratura curvarum* (On the quadrature of curves) [Newton, Quadrature][10] was published for the first time as appendix of *Opticks* [Newton, Opticks] in 1704 and republished in 1711. In the comprehensive treatise on the new method entitled *On the method of fluxions* (set in work in 1664 and finished in 1671, published in 1736, a translation into French was published in 1740) [Newton, Method of Fluxions] Newton formulated two correlated problems to be solved.

a. Calculate the relation between fluxions if the relation between their fluents is given. (M1)
b. Calculate the relation between the fluents if the relation between their fluxions is given. (M2)[11]

Following Newton and using the concept of function which had been later introduced by Leibniz, Johann Bernoulli and Euler [Thiele 1999], [Thiele 2007] fluents and fluxions are in general functions of time. In terms of mechanics, the fluents may be interpreted as the subsequent positions or distances x, i.e. the *paths* of bodies travelled through by motion, whereas the fluxions are related to the velocities. The momentary increment of distance is represented by the product of velocity $\dot{x}$ and an infinitesimal quantity o, i.e. the quantity $\dot{x} \cdot o$ which has to be added to the position occupied at a certain time, $x + \dot{x} \cdot o$. Therefore, the later program for mechanics had to be finally adapted to an existing *complete* program and new method for mathematics and vice versa, the program for mathematics had to be adapted to the different versions of the program for mechanics.

Newton introduced two types of quantities nowadays called variables and constants which are distinguished by their numerical values being either indeterminate or determinate, respectively. The fluents are increased (augmented) *gradually* and

[10] "The *Introductio ad Quadraturum Curvarum* is the introduction that Newton wrote to one of two mathematical treatises appended to the first edition of his *Opticks*, published in 1704. These mathematical treatises were republished in 1711, in *Analysis per Quantitatum Series, Fluxiones, ac Differentias, cum Enumeratione Linearum Tertii Ordinis*, edited by William Jones. The Latin text available here is taken from this edition of 1711. Also available is a translation into English made by John Harris and published in the second volume of his *Lexicon Technicum*, published in 1710." [http://www.maths.tcd.ie/pub/HistMath/People/Newton/Quadratura/]

[11] Following Euler, the subject of the calculus is the geometric ratio between different infinitesimal quantities. "In calculo autem infinite parvorum nil aliud agitur, nisi ut ratio geometrica inter varia infinite parva indagetur." [Euler E212, § 85] (compare Chap. 5)

indefinitely, i.e. continually,[12] but are not augmented by constant increments or indivisible least elements as it had been proposed by Cavalieri [Cavalieri].[13]

Looking at the time line of the publication of Newton's and Leibniz's papers on the calculus it is easily confirmed that most of the work, beside the difficult understanding of the published part, had to be devoted to the reconstruction and recovery of the missing unpublished parts. The most famous and simultaneously the most successful and controversially discussed reconstruction of Newton's method had been done by Leibniz. Recently, the historical background had been summarized under the title *Equivalence and priority* [Meli]. Nevertheless, there is an essential internal difference which prevents to establish a full equivalence between Newton's and Leibniz's methods as far as the application in mathematics and mechanics is concerned. There is no doubt that Leibniz's representation is more convenient for the formulation and solution of problems including the interpretation of results [Euler E212, §§ 114–118].[14] Finally, Leibniz's invention of the calculus is centred upon order [Leibniz, quadrature arithmétique], especially the order represented by the set of integers.[15]

The treatises and pamphlets written by Newton, Leibniz, Johann Bernoulli and other authors in the dispute on the priority are valuable summaries of the development of the new method and the relation between the two representations, (i) the report of the commission of the Roal Society *Commercium Epistolicum D. Johannis Collinii* (1712) [Newton (Collins), Commercium], (ii) Newton's *Account* and (iii) the republication of the *Commercium* in 1724. The *Commercium* had been carefully composed by Newton [Djerassi, Calculus]. Newton already accentuated the main topic of the controversy by correlating and opposing in the title the *Method of Fluxions* and the *Differential method*.

[12] "LX. J'appellerai *Quantités Fluentes*, ou simplement *Fluentes* ces Quantités que je considéré comme augmentées graduellement & indéfiniment, (...) pour les distinguer des autres quantités qui dans les Equations sont considérées comme connues & déterminées qu'on représente par les Lettres initiales *a,b,c* &c. & je représenterai par les mêmes dernieres Lettres surmontées d'un point $\dot{v}, \dot{x}, \dot{y}$ & $\dot{z}$ les vitesses dont les Fluentes sont augmentées par le mouvement qui les produit, & que par conséquent on peut appeler *Fluxions*. Ainsi pour la Vitesse ou Fluxion (...)." [Newton, Method of Fluxions]

[13] "Cavalieri made a rational systematization of the method of indivisibles. His view of the indivisibles gave mathematicians a deeper conception of sets: it is not necessary that the elements of a set be assigned or assignable; rather it suffices that a precise criterion exist for determining whether or not an element belongs to the set." [http://galileo.rice.edu/Catalog/NewFiles/cavaleri.html]

[14] "116. It might be uncivil to argue with the English about the use of word and a definition, and we might easily be defeated in a judgement about the purity of Latin and the adequacy of expression, but there is no doubt that we have won the prize from the English when it is a question of notation. For differentials, which they call fluxions, they use dots above the letters. (...) if many dots are required, much confusion and even more inconvenience may result." [Euler E212, § 116]

[15] Euler generalized Leibniz's procedure [Euler E212, § 85] and correlated the order of natural numbers 1,2,3,4,... or $1 < 2 < 3 < 4 < \ldots$ to the order of differentials $dx > dx^2 > dx^3 > dx^4 > \ldots$ which form the inverse counterpart. Hence, it may be concluded that the differentials can be treated like numbers. The latter order is invariant against the augmentation or diminution of the differentials due to multiplication by any finite number $a, b, c, d, \ldots$, i.e. the relation $adx > cdx^2 > bdx^3 > ddx^4 > \ldots$ holds independently of the mutual relations between $a, b, c, d, \ldots$ [Euler E212, § 97]. (Compare Chap. 5)

The argumentation is the same as in the first Edition of the *Principia* where Newton commented that the only difference between his own and Leibniz's method is to be found in the different notation, i.e. the use of symbols which indicates the priority in invention by Newton.[16] In this comment, Newton made no mention of any difference in the methods nor described his approach as the Method of Fluxions. These words only appeared in the codified part of the letter sent to Leibniz in 1678: "Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; & vice versa." [Newton, Letter to Leibniz].[17] Newton treated the following problems, first to demonstrate the priority in the invention of the calculus, second to compare the results he had obtained to the results of Leibniz and, third to demonstrate the advantage, i.e. the easier applicability and the elangance of his Method of Fluxions.

3.1 Newton's Concept of Fluents and Fluxions

In the first announcements of the new method by Newton and Leibniz published in 1687 and 1684, respectively, both authors distinguished between an *arithmetical* and a *geometrical* part of the foundation. Newton embedded the presentation of the calculus in the famous treatise entitled *Philosophiae naturalis principia mathematica* [Newton, Principia, Book II, Sect. II, Lemma II] published in 1687 whereas Leibniz presented an as famous paper entitled *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus* [Leibniz, Nova methodus] in 1684 taking no refer-

[16] "In Briefen, die ich vor 10 Jahren mit dem sehr gelehrten Mathematiker *G. W. Leibniz* wechselte, zeigte ich demselben an, daß ich mich im Besitze einer Methode befände, nach der man Maxima und Minima, Tangenten ziehen und ähnlich Aufgaben lösen könne, und zwar lasse sie sich ebensogut auf irrationale wie auf rationale Größen anwenden. Indem ich die Buchstaben der Worte, die meine Meinung aussprachen, versetzte, verbarg ich dieselben. Der berühmte Mann antwortete mir darauf, er sei auf eine Methode derselben Art verfallen, die er mir mitteilte, und die von der meinigen kaum weiter abwich als in der Form der Worte und Zeichen." [Newton, Principia, 1687, 1713 and 1726].

[17] "Eventually Oldenburg persuaded Newton to write to Leibniz on the pretext that Leibniz had some mathematical queries that only he, Newton, could answer. Reluctantly, Newton wrote two major letters through Oldenburg to Leibniz. The so-called *First Letter* – written in June 1676 – was 11 pages long and the so-called *Later Letter* – written in October 1676 – was 19 pages long. Together, they summarized Newton's mathematical discoveries and were designed to show Leibniz that he had made a number of breakthroughs many years ago. But even then, nervous that others would steal his ideas, Newton did not mention the calculus specifically; instead, he added a coded sentence (...). He wrote 'I cannot proceed with the explanation of the fluxions [the calculus] now, I have preferred to conceal it thus: 6accdae13eff7i3l9n4o4qrr4s8t12vx.' This encryption defined the meaning of the calculus (...). 'Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire: et vice versa.' *Given an equation involving any number of fluent quantities, to find the fluxions: and vice versa*." [http://courses.science.fau.edu/~rjordan/phy1931/NEWTON/newton.htm]

ence to any subject outside arithmetic and geometry.[18] Newton modelled the generation of an increment or decrement of a quantity by moments which are themselves generated by multiplication of a velocity and a time element. The change of the quantities is related to a flux or to their changes in time.

> Has quantitates ut indeterminatas & instabiles, & quasi motu fluxuve perpetuo crescentes vel decrescentes hic considero, & eorum incrementa vel decrementa momentanea sub nomine momentorum intelligo: ita ut incrementa pro momentis addititiis seu affirmativis, ac decrementa pro subductitiis seu negativis habeantur. Cave tamen intellexeris particulas finitas. [Newton, Principia, Book II, Sect. II, Lemma II.]

The quantities under consideration being "increased or decreased by perpetual motion" are denoted by $A, B, C, \&c.$ which may geometrically interpreted as the sides A and B of a rectangle.

> Wherefore the sense of the Lemma is that if the moments of any quantities A, B, C, &c. increasing or decreasing by a perpetual flux, or the velocities of the mutation which are proportional to them, be called a, b, c, &c. the moment or mutation of the generated rectangle AB will be aB + bA; (...). [Newton, Principia, p. 250]

In the early treatise on *Method of Fluxion* from 1671, Newton mainly gave the same foundation of the rules by the same *arithmetical* principles without a direct reference to a geometrical model or interpretation in the same manner as Leibniz presented his version in *Nova methodus*. The set of basic notions is composed of the following terms called fluents, fluxions and moments labelled by $x, y, z, \ldots,\ \dot{x}, \dot{y}, \dot{z}, \ldots$ and $\dot{x} \cdot o, \dot{y} \cdot o, \dot{z} \cdot o, \ldots$, respectively. The fluents are increased by the moments.

> XV. Puis donc que les moments comme $\dot{x} \cdot o$ & $\dot{y} \cdot o$ sont les accessions ou augmentations indéfiniment petits des Quantités Fluentes x & y (...) après un intervalle indéfiniment petit de tems, il suit que ces Quantités x & y après une intervalle indéfiniment petit de tems, deviennent $x + \dot{x} \cdot o$ & $y + \dot{y} \cdot o$, & par conséquent l'Equation qui en tout tems exprime également la Relation des Quantités Fluentes, exprimera la Relation entre $x + \dot{x} \cdot o$ & $y + \dot{y} \cdot o$ (...) ainsi on peut substituer dans la même Equation $x + \dot{x} \cdot o$ & $y + \dot{y} \cdot o$, au lieu x & y. [Newton, Method of Fluxions, p. 25]

The arithmetical operations addition and subtraction for finite quantities are transferred to the arithmetical operations for *moments*. Arithmetically, the moment is an increment or decrement of a fluent, $x + m_x = x + \dot{x} \cdot o$ & $y + m_y = y + \dot{y} \cdot o$. The moments are composed of quantities of different kind as the product of a *fluxion* and a certain *time element*. Newton assumed that there is an "infinitely time o" beside any other time elements of finite length. These time element seems to be appropriate to describe the *transition from rest to motion* or *from motion to rest* in term of

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[18] In 1680, Leibniz composed a paper entitled "Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum, solidorum, aliisque communem calculum transcendentibus" ("The elements of the new calculus for differences and sums, tangents and quadratures, maxima and minima, dimensions of lines, surfaces, and solids, and for other things that transcend other means of calculation") which had been only published in the 19th century [Leibniz, Gerhardt, Elementa], [Leibniz (Child)]. Here, Leibniz accentuated the arithmetical operations of *addition* and *subtraction* ("new calculus for differences and sums"), i.e. those *arithmetical operations* which had been excluded by Newton (see Sect. 3.1.1).

nascent and evanescent motion, respectively. Newton stressed the *generation* of motion described in terms of an "infinitely little time o" whereas Leibniz emphasized the *evanescence* of motion. However, Newton claimed that one has always to study the *ratios* of different fluents and different fluxions, but not the increments or decrements of a single fluent. Hence, either the relation between the fluents or the relation between the fluxions should be given. Then, it is quite natural to define the subject of the calculus as the investigation of the *geometric* ratio of the fluents compared to the *geometric* ratio of the fluxions (compare Chap. 5).[19] Following Newton and using the concept of function which had been later introduced by Leibniz, Johann Bernoulli and Euler [Thiele 1999], [Thiele 2007] fluents and fluxions are in general functions of time. In the Method of Fluxions, the *explicit* time dependence is given by the "infinitely little quantity o". Newton assumed that the fluents and fluxions are independent of the quantity o, $X(t,o) = x(t) + m_x(t,o)$, whereas the *moments* depend on time, and the increment of time, $m_x(t,o) = \dot{x}(t) \cdot o$. Hence, the *increment* of time is supposed to be *independent* of the time.[20] The variable t can be arbitrarily chosen $X(t,o) = x(t) + m_x(t,o) = x(t) + \dot{x}(t) \cdot o$. The *analytical* expression for $X(t,o)$ is *invariant*, therefore, all other fluents and fluxions obey not only the same time variable and the same "infinitely little quantity o", but also the same analytical relations between fluents, fluxions and moments.

3.1.1 The Arithmetic and Geometric Representation of Quantities

> Genitam voco quantitatem omnem quae ex Terminus quibuscunque in Arithmetica per multiplicationem, divisionem & extractionem radicum; in Geometria per inventionem vel contentorum & laterum, vel extremarum & mediarum proportionalium absque; additione & subductione generatur. [Newton Principia (1st ed. 1687), Book II, Sect. II, Lemma II, p. 250][21]

[19] "Quantitates, ut & quantitatum rationes, quae ad aequalitatem dato tempore constanter tendunt & eo pacto propius ad invicem accedere possunt quam pro data quavis differentia; fiunt ultimo aequales." [Newton, Principia (1st ed. 1687) Book I, Lemma I]

"Quantitates, ut & quantitatum rationes, quae ad aequalitatem tempore quovis finito constanter tendunt & ante finem temporis illius proprius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo aequales." [Newton, Principia (2nd ed. 1713)]

"Les quantités & les raisons des quantités qui tendent continuellement à devenir égales pendant un temps fini, & qui avant la fin de ce temps approchent tellement de l'égalité, que leur différence est plus petite qu'aucune différence donnée, deviennent à la fin égales." [Newton, Principa, Book I, Lemma I (Châtelet)]

[20] Later, Euler made use of this idea and postulated that the "increment of velocity is independent of velocity". Apparet haec incrementa celeritatis non pendere ab ipsa celeritate c, sed eundem habitura esse valorem, quantumvis magna aut parva ponatur c." [Euler E015, § 131] (compare Chap. 4) Here, we find the true origin of one of the basic principles of classical mechanics.

[21] "J'appelle quantité produite toute quantité formée sans addition & sans soustraction, soit arithmétiquement par la multiplication, la division, ou l'extraction des racines de quantités simples, ou de leurs puissances, soit géométriquement par la détermination des produits & des racine, ou des extrêmes & des moyens proportionnels." [Newton, Principia, 1st ed. 1687, Book II, Sect. II, Lemma II, (Châtelet)] "I call any quantity a *Genitum*, which is not made by addition or subtraction of diverse parts, but is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever; in geometry by the invention of contents and

The idea of a permanent flux had been incorporated in mathematics and mechanics by Newton for the representation of increments and decrements of "indeterminate and unstable" or, using Euler's terminology [Euler E101], variable quantities. Referring to the *permanent* or *persistent* flux, all attempts to introduce any other kind of quantities being not permanently flowing is automatically and intentionally excluded.[22] Moreover, Leibniz's procedure to explain the differentials as evanescent *differences* of variables [Leibniz, Nova methodus] or Cavalieri's indivisibles had been replaced by the generation of infinitesimal moments by *multiplication* of a finite velocity by an "infinitely little quantity o" (compare Chap. 5 for Euler's procedure and Euler's confidence in the general validity or arithmetical operations addition, subtraction, multiplication and division [Euler E212], [Euler E387], [Euler E017]). Newton's procedure may be interpreted as a reaction to Leibniz's *Nova methodus* published in 1684 [Leibniz, Nova methodus]. The complete set of algorithms for operations with infinitesimal, finite and infinite quantities had been only analyzed by Euler [Euler 1727], [Euler E212] who distinguished consequently between arithmetic and geometric ratios of any kind of numbers.

The mechanical standard model of such quantities was introduced by Newton who assumed that the fluxions are related to velocities. Newton assumed time as basic concept for the introduction of infinitesimal quantities. The time element o is introduced as a term constituting the product $\dot{x} \cdot o$ of a velocity $\dot{x}$ and the time element o. Following Newton, these quantities are to be considered as *mathematical* quantities which are of different kind being either (i) *points* or (ii) *lines, surfaces* and *solids*. Lines, surfaces and solids are generated by a continual motion whereas points are not generated by a continual motion. Usually, the objects of set (ii) are considered as *extended* things whereas the point is assumed to be having no *extension*. Therefore, Newton's construction of "Nature every Day seen in the motion of Bodies" is different from Descartes' definition of the body as an "extended thing" or "res extensa sive corpus". Following Newton, the extension is not given as a basic invariant property of natural things, but is generated either by nature or modelled *geometrically* by the motion of points, the motion of lines and the motion of surfaces. The generation of points by any kind of motion is not considered. In agreement with Euclid's foundation of geometry,[23] Newton accepted the "point" as basic

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sides, or of the extrema and means of proportions. Quantities of this kind are products, quotients, rectangles, surfaces, squares, cubes, square and cubic sides and the like." [Newton, Principia, Book II, Lemma II, p. 256]

Newton excluded addition and subtraction or increments and decrements by means of Cavalieri's indivisibles including Leibniz's differentials.

[22] Is this a response to Leibniz's *Brevis demonstratio*? Here, Leibniz compared the motion of two non-interacting bodies which interact with the same third, the earth. The parameters the bodies are distinguished from each other are the heavy masses of different magnitude (weights). The conservation of living forces in the *impact of two bodies* had been later studied by Leibniz [Leibniz, Specimen UV (11)] Here, the parameters are the *inert masses*. Hence, this interaction is independent of a third body. The living forces are conserved. A e e + B f f = A (e) (e) + B (f) (f) = const. [Leibniz, Specimen UV (11)] Leibniz did not discuss the differences (e) – e and (f) – f.

[23] Following Euclid, the point does not consist of parts. "Definition 1. A *point* is that which has no part. Definition 2. A *line* is breadthless length. Definition 3. The ends of a line are points. Definition 4. A *straight line* is a line which lies evenly with the points on itself. Definition 5.

notion and added the "time" as additional basic notion. Then, following Newton, the other Euclidean geometrical objects, the line, the surface and the solid, are not given, but generated by the fundamental process being the *mathematical* representation of that *phenomenon* "every Day seen in the motion of Bodies". Therefore, we can conclude: As far as there are no leaps in the motion of bodies seen every day it follows that lines are created by a continual motion of points. In contrast to the *Leibnizian principle of continuity* which is transferred from geometry into mechanics, the *Newtonian principle of continuity* is represented by the *continual flow of time*, therefore, the continuity is transferred from *mechanics into geometry* (here, we consider "time" as belonging to mechanics which is only justified if time is assumed as "absolute time" [Newton, Principia]), as far as lines, surfaces and solids are concerned, and to mathematics, as far as *infinitesimal* quantities are concerned. Hence, after the invention of the calculus by Newton and Leibniz, the same problem arises for the transfer of the principles of the calculus to mechanics. However, even for the inventors of the calculus, this step can be taken by no means for granted since it was almost automatically necessary to accept the earlier Cartesian program and basic principles of Cartesian methodology.[24]

Euler and generalized the Cartesian program. Analyzing Euler's philosophical statements people claimed that Euler renewed the Cartesian dualism between body and soul. However, comparing Euler's basic assumptions on the nature of bodies with the original Cartesian version, a remarkable and strong difference can be observed. Euler considerably modified Descartes concept of bodies by the invention of bodies of infinitesimal magnitude.

Descartes claimed that the *analytical representation* of the curve by a formula depends on the choice of abscissa (representing the independent variable) and the ordinate (representing the dependent variable).[25] The calculus is usually considered as a mathematical theory which is closely related to geometry. However, the application of the calculus was mathematically and mechanically from the very beginning since the *Principia* are based essentially on the new method developed by Newton. Therefore, the introduction (as well as the interpretation) of infinitesimal quantities

A *surface* is that which has length and breadth only. Definition 6. The edges of a surface are lines. Definition 7. A *plane surface* is a surface which lies evenly with the straight lines on itself. Definition 8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line." [Euclid, Elements, Book I] Following Euclid's order of definitions, at first, although point and line had been introduced as different objects they are compared to each other by their common properties, the breadthless, at second point and line are compared to surface and vice versa. Newton removed the Euclidean order between "point" and "line" assuming that the "line is generated by the motion of a point". Therefore, Newton replaced "point, line, surface and solid" with "point and time" or "the continual motion of a point" which generates a line. By Def. 3, Euclid introduced a finite line which is not closed, but has two ends being points. Leibniz stated that the ends of the line are different from isolated points since they are the border of the line [Leibniz, Initia, pp. 357–358]. Therefore, the Newtonian time is a *finite* quantity since the result of the motion of a point is always a line of *finite* length.

[24] "Men of recent times, eager to add to the discoveries of the Ancients, have united the arithmetic of variable with geometry." [Newton, Math 4:421] (compare Chap. 1)

[25] This indeterminacy in the relation between x and y played an important role in the analysis and the interpretation and foundation of the calculus by Leibniz (compare [Bos]).

was motivated and challenged by the mathematical and mechanical problems. From the very beginning up to present time, the same distinction had also been preserved for the interpretation of the *differentials*. Independent of the various names which had been chosen for these objects by different authors like fluxions (Newton), indivisibles (Newton, Leibniz), differentials, infinitesimal quantities, spatiolum, tempusculum or differences, the basic features of the methodological background had been preserved during the last three centuries. Newton's and Leibniz's theory can be traced back to the ancients, especially to Euclid (about 325–265) and Archimedes (287–212), who formed the prototypes for the mathematical and mechanical sciences, i.e. geometry and statics, respectively. The general validity of Euclid's axioms for geometry had been only questioned in the 19th century by Gauß, Bolyai and Riemann. The new science of mechanics had been based on Archimedes' laws for statics. The science of motion is a novelty which had mainly been introduced by Copernicus (1473–1543), Kepler (1571–1630),[26] Bacon (1561–1626), Galileo (1564–1642)[27] and developed by Descartes (1596–1650),[28] Huygens (1629–1695) and Newton (1643–1726).

3.1.2 One Universal Infinitesimal Quantity

Leibniz studied the relations between two quantities represented by the variables y and x and discussed the following situations (i) x is given and y is asked and (ii) y is given and x is asked. Hence, there is an arbitrariness of the choice of the variables and their differentials dy and dx which was equivalent to the "freedom of the choice on the nature of the approximating polygons, or rather in the progression of variables" [Bos, p. 90], [Bos, Leibniz].

> The fact that the variables in the Leibnizian calculus are not functions should be stressed; it is one of the main aspects in which that calculus differs from the forms of the infinitesimal calculus developed later. Indeed, when later the concept of functions acquired a central place in the calculus, the problem of the indeterminacy of the differentials vanished. [Bos, p. 91]

In Newton's Method of Fluxions, this problem cannot occur since Newton has "only one infinitely little quantity o" [Newton, Quadrature (Harris)].

Following Newton, all mechanical quantities are represented by fluents being time related quantities by name and construction and, consequently, as functions of

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[26] A relation between the motion of planets and numbers and polyhedrons (five regular Platonic bodies) had been established by Kepler.

[27] Galileo demonstrated that the path of a thrown body is a geometrical curve, a parabola [Galileo, Discorsi]. Before Galileo the paths were also related to geometry ad considered either as straight lines or circles.

[28] Descartes introduced (i) rest and (ii) motion along a straight line in the same direction as basic states of all bodies which are not disturbed by other bodies.

time.[29] The fluxions describe the velocity the fluent is changing its value in time. There is either an increment or a decrement. Quantities of this type may be represented *mechanically* by the change of the place in dependence of time, called motion of the body, which is mathematically represented by two quantities related to (i) coordinates and (ii) time, $s = s(t) \neq \text{const}$, whereas the resting of a body is only expressed in terms of coordinates, $s = \text{const}$. In case of rest "time" is not "defined" by a relation mediated by a finite value of "motion", but by the absence of motion described by the independence of time. Using the Leibnizian representation of the calculus, from the relation $s = \text{const}$, two equivalent representation (a) $d.s = ds = 0$ and (b) $\Delta.s = \Delta s = 0$ follow for the adequate representation[30] of the independence of time assuming that the time dependence may be expressed in terms of coordinates.

Representing different types of changes by (a) and (b), the *time independence* is also independent of the representation, but is a general feature of the considered body or the considered system of bodies.[31] The problem arises due to the representation of the time-independence by algebraic relations between spatial and temporal intervals: (a1) $d.s = ds = v \cdot dt$ and (b1) $\Delta.s = \Delta s = v \cdot \Delta t$. Assuming the homogeneity for the variables of the equation [Leibniz, Homogeneity], the time variable has mathematically to be treated by the same operations which had been applied to coordinates. The operation applied to coordinates $d.s = ds$ has to be transferred to time variable $d.t = dt$ to ensure the homogeneity of the terms the equation is made up. Following Newton, the time is represented by *two universal* quantities being either finite, then time is called absolute time, or infinitesimal o, then o is the only infinitesimal quantity [Newton, Method of Fluxions], [Newton (Collins), Commercium], $d.t = dt = o$, or, $d.t_{\text{absol}} = dt = o_{\text{univer}}$, or, $o_{\text{universal}} = dt_{\text{Newton}}$. However, Newton stressed that these quantities $o_{\text{universal}} = dt_{\text{Newton}}$ are essentially different from the indivisibles introduced by Cavalieri and Leibniz (since the geometrical objects are generated by a continual motion). Therefore, Newton did not represent the change in motion by "acceleration" (defined by $dds/dtdt = d^2s/dt^2$) we are now accustomed to, but by the relation between "errors" dds and times and forces. As it had been discussed in Chap. 2, Newton obtained a relation between the impressed moving force and the errors, i.e. the increments of the increments of the path.

> Cor. 2. But the errors $(\ldots)$ are as the forces and the squares of the times conjunctly. [Newton, Principia, Book I, Sect. I, Lemma IX]

Analytically, the relations are expressed as follows, $s^{\text{errors}}_{\text{begin}} \sim K \cdot t^2_{\text{begin}}$, $dds \sim K \cdot dt^2$, and become only equalities if the mass is added as an additional parameter called *inert mass*. The basic assumption is that the inert mass resists the change of the state caused by the impressed moving force [Euler E842].

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[29] Using the notion of a function, the fluent is represented as a function of time, $f = f(t)$. The notion of function had been implicitly assumed by Newton and Leibniz and later explicitly introduced by Euler and the Bernoullis [Thiele 2007].

[30] This representation is independent of the interpretation of *differentials* as infinitesimal quantities.

[31] "1. Motus est translatio corporis ex loco, quem occubabat, in alium. Quies vero est permansio corporis in eodem loco." [Euler E015/016, § 1]

$$s_{\text{begin}}^{\text{errors}} \sim K_{\text{impress}} \quad \text{and} \quad s_{\text{begin}}^{\text{errors}} \sim 1/m_{\text{inert}} \tag{3.1}$$

Comparing two bodies to each other, then, independently of time the deviations, the "errors", are smaller if the mass is greater. The whole problem is described by four quantities which can be divided into two sets. Set 1 comprises "errors and times" and is related to the position and the change of position, i.e. *s, ds, dds* and t, dt, dt^2, whereas set 2 comprises the mass m_{inert} and the forces K_{impress}.

3.2 Newton's Algorithm: Method of Fluxions

Newton assumed that the relation between two fluents denoted by x and y [Newton, Method of Fluxions], [Newton, Quadrature (Harris)],[32] [Kowalewski] is given by the analytical expression

$$x^3 - ax^2 - axy - y^3 = 0. \tag{3.2}$$

The corresponding fluxions are denoted by $\dot{x}$, $\dot{y}$. Assuming that the fluents are given, the ratio of the fluxions is to be calculated. The result is

$$\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - ax}{3x^2 - 2ax + ay}. \tag{3.3}$$

It is instructive to analyse Newton's procedure step by step. Consider an analytical expression depending on some constants and two different kinds of variables

$$f(A,x,y) = A + a \cdot x + b \cdot x^2 + c \cdot y + e \cdot y^2 = 0. \tag{3.4}$$

Following Newton, the fluxion is calculated by adding the moments to the fluents. Later in 1715, these quantities had been called increments or decrements [Taylor, Methodus]. The moments are defined as the product of the velocity the fluents are changing multiplied by a quantity having the dimension of a time, the "infinitely little quantity o". Newton insisted that this increment is different from any of Cavalieri's indivisibles since any of the objects, e.g. lines, surfaces or solids, are generated by motion and are not composed by simultaneously existing (coexisting) parts of any geometrical object. Moreover, Newton referred to the Ancients to demonstrate that his method is of the same mathematical rigor as the methods of the Ancients. However, Newton cannot not exclude *arithmetical* operations which are crucial for the new method, but went beyond the fundamentals the Ancients had been based mathematics upon. This concerns the concept of infinity. Already assuming an infinite system of infinitesimal quantities, $o, o^2, o^3, \ldots$, Newton necessarily provoked the question how the counterpart of the infinitesimal, the infinite, may be

[32] "Fluxiones sunt quam proxime ut Fluentium augmenta aequalibus temporis particulis quam minimis genita; & ut accurate loquar, sunt in prima ratione augmentorum nascentium; exponi autem possunt per lineas quascunque quae sunt ipsis proportionales." [Newton, Quadrature (Harris)]

treated arithmetically. Following Cavalieri and other authors, the geometrical object can be regarded to be made up of an infinite number of infinitesimal indivisible parts.[33]

Newton did not consider the relations between the object and its part, but the beginning and the end of the generation of lines, surfaces and solids. Replacing the fluents with fluents and their increments, it follows

$$\begin{aligned} f(A,x+\dot{x}\cdot o,y+\dot{y}\cdot o) &= A+a\cdot(x+\dot{x}\cdot o)+b\cdot(x+\dot{x}\cdot o)^2 \\ &+c\cdot(y+\dot{y}\cdot o)+e\cdot(y+\dot{y}\cdot o)^2 \end{aligned} \tag{3.5}$$

In the next step, the result of the calculation is ordered according to the different magnitude of the terms.

$$\begin{aligned} &= A+a\cdot x+b\cdot x^2+c\cdot y+e\cdot y^2 \\ &(a\cdot\dot{x}+c\cdot\dot{y}+2\cdot b\cdot x\cdot\dot{x}+2\cdot c\cdot y\cdot\dot{y})\cdot o+(b\cdot\dot{x}^2+c\cdot\dot{y}^2)\cdot o^2 \end{aligned} \tag{3.6}$$

Following Newton, the magnitude of the terms depends only on the magnitude of the powers $1,o^1,o^2,o^3,\ldots.$ if all terms in the brackets are finite. Then, after the division by o, the relation

$$(a\cdot\dot{x}+c\cdot\dot{y}+2\cdot b\cdot x\cdot\dot{x}+2\cdot c\cdot y\cdot\dot{y})\cdot 1+(b\cdot\dot{x}^2+c\cdot\dot{y}^2)\cdot o=0. \tag{3.7}$$

is obtained. The second term is infinitely small with respect to the first term and can be omitted.[34] Therefore, the basic equation for the ratio of fluxions is derived from the term in the first bracket

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[33] Cavalieri, *Geometria indivisibilibus continuorum nova quadam ratione promota*. Also Cavalieri justified his method by the reference to the Ancients. “Die Gültigkeit der Methode wurde besonders von seinem Zeitgenossen, dem Jesuiten Guldin angezweifelt und angegriffen, der eigene Oberflächenberechnungen durch Schwerpunktbetrachtungen anstellte, aus denen die Guldinschen Regeln hergeleitet werden. Noch 1647 (*) rechtfertigte Cavalieri seine Methode u.a. mit Hinweisen auf ähnliche Gedanken beim letzten großen griechischen Mathematiker, Pappos von Alexandria (um 33 n.Chr.).” (*) [Cavalieri, Exercitationes Geometricae] [http://www.learnetix.de/learnetix/mathe/nachschlagen/nach_bm/nach_bm_cavalieri.html]
“The principle of indivisibles had been used by Kepler in 1604 and 1615 in a somewhat crude form. It was first stated by Cavalieri in 1629, but he did not publish his results till 1635. In his early enunciation of the principle in 1635 Cavalieri asserted that a line was made up of an infinite number of points (each without magnitude), a surface of infinite number of lines (each without breadth), and a volume of an infinite number of surfaces (each without thickness). To meet the objections of Guldinus and others, the statement was recast, and in its final form as used by the mathematicians of the seventeenth century it was published in Cavalieri’s *Exercitationes Geometricae* in 1647; the third exercise is devoted to a defence of the theory. This book contains the earliest demonstration of the properties of Pappus. Cavalieri’s works on indivisibles were reissued with his later corrections in 1653.” [http://www.maths.tcd.ie/pub/HistMath/People/Cavalieri/RouseBall/RB_Cavalieri.html]
The space is composed of indivisibile elements, approximations are only of minor importance [Klein, Elementarmathematik, p. 231]

[34] One of the main topics in the discussion on the foundation of the calculus was whether this omission is an approximation or a rigorous result (compare Chap. 5)

$$(a \cdot \dot{x} + c \cdot \dot{y} + 2 \cdot b \cdot x \cdot \dot{x} + 2 \cdot e \cdot y \cdot \dot{y}) = 0. \tag{3.8}$$

Here, Newton implicitly assumed that the *geometric ratio* of two infinitesimal quantities is not only determined, but can be expressed by a finite quantity, if both the infinitesimal quantities are *equal* to each other, i.e. $o/o = 1$. The result is

$$\frac{\dot{x}}{\dot{y}} = -\frac{c + 2 \cdot e \cdot y}{a + 2 \cdot b \cdot x} \tag{3.9}$$

Newton never assumed that the "infinitely little quantity o" is equal to zero. Then, the expression in the brackets is zero and obtained a relation between the finite fluents and finite fluxions.[35] All terms of infinitesimal magnitude are removed since they appear only temporarily in the reckoning procedure. Therefore, Leibniz interpreted the differentials as *fictitious* quantities (compare the comment by Cantor [Cantor] which will be discussed in Chap. 5).

In the following step, Newton's model will be mechanically interpreted. It is assumed that the relation is not modified by the increments of the fluents, i.e. the augmentation of diminution of $f(A, x, y)$ is not of finite magnitude, but equal to zero.

Newton's model can be successfully applied for the motion of a body which is confined to a plane. In the 2D version, a motion is performed in the x-y-plane and the tangents of the curved line representing the path of the body are given by the geometric ratio $\dot{x}/\dot{y}$ where both quantities are functions of time. In Leibniz's notation, the result is expressed in terms of the geometric ratio of differentials (i) *dx/dy* and (ii) *dy/dx* (compare Sect. 3.3). Moreover, in contrast to Newton's result, there are two possibilities to combine functions and independent variables, (i) $x = x(y)$ and (ii) $y = y(x)$ [Bos]. A common variable like time is *not* included, but can be additionally introduced. Following Leibniz, in contrast to Newton's approach, neither the independent variables nor the function are to be mechanically interpreted, but are purely mathematical quantities. This procedure had been generalized by Taylor [Taylor, Methodus] and Euler [Euler E212].

Newton's algorithm can be readily interpreted in terms of Leibniz's living forces which are represented by the product of the mass and the square of velocity $m \cdot v^2$. Following Descartes and Newton, motion is given by the product of mass and velocity $p = m \cdot v$. Assuming $a = 0,\ c = 0$ and specifying the other parameters to be the force constant $k/2$ and the reciprocal mass $1/2m$, the relation for the ratio $\dot{p}/\dot{x}$ between the fluxion of momentum $\dot{p}$ and the fluxion of position $\dot{x}$ can be calculated if a relation between the fluents p and x is known. Following Newton, the change of motion is proportional to the impressed moving force and necessarily given by the *fluxion* of the product of mass and velocity, i.e. $\dot{p} = m \cdot \dot{v}$.

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[35] Therefore, Newton assumed implicitly that the "time element" o is different from zero. Following Newton [Newton (Collins), Commercium], there is only one infinitesimal quantity, i.e. only one and the same infinitesimal quantity. This quantity is of the dimension of a time, i.e. it is *not* a purely mathematical quantity, but related to the motion and the generation of motion as well as to the *generation of geometrical objects* [Newton, Quadrature (Harris)], the generation or creation is not by the "apposition of parts", but by a continual motion where parts can be internally neither *distinguished* nor *assigned*. Any distinction of parts is due to an *external* operation.

Assuming that the relation between the fluents of path and motion is invariant $E = b \cdot x^2 + e \cdot p^2 = \text{const}$, then it is only valid for certain pairs of positions and velocities which fulfil for any time and any increment of the fluents the relation

$$f(E,x,y) = E - b \cdot x^2 - e \cdot p^2 = 0. \tag{3.10}$$

Mechanically interpreted, Newton's exemplification of the method of fluxions straightforwardly results in a certain kind of those conservation laws whose mechanical interpretation had not been regarded as meaningful by Newton.[36] For $E = 0$, it follows $b \cdot x^2 + e \cdot p^2 = 0$ and the relation is only valid for the pair $x = 0$ and $y = 0$. Hence, for $E \neq 0$, it necessarily holds $b \cdot x^2 + e \cdot p^2 \neq 0$ and the relation is only valid for any pair (i) $x_i \neq 0$ and $p_i \neq 0$, (ii) $x = 0$ and $p_{j \neq i} \neq 0$ with $E = e \cdot p_j^2$ and (iii) $x_{k \neq i} \neq 0$ and $p = 0$ with $E = b \cdot x_k^2$. Then, it follows that Newton implicitly assumed that the relation depending on moments fulfils the same requirements as supposed to be valid for the fluents.[37]

$$f(E,x,y;m_x,m_y) = E - b \cdot (x + m_x)^2 - e \cdot (p + m_p)^2 = 0, \tag{3.11}$$

$$k \cdot x \cdot \dot{x} + \frac{1}{m} \cdot p \cdot \dot{p} = 0, \tag{3.12}$$

$$k \cdot (m \cdot \dot{x}) \cdot x + p \cdot \dot{p} = 0, \tag{3.13}$$

$$k \cdot x + \dot{p} = 0. \tag{3.14}$$

Finally, the well-known equation of motion

$$\dot{p} = -k \cdot x \tag{3.15}$$

is obtained. Although the relation between fluents and fluxions had been already determined by Eq. (3.12) its further treatment beyond the supposed frame is of special interest. The result is the equation of motion whose general form had been postulated by Newton by the 2nd Law, but never analytically formulated. However, after the reduction of the relation to the equation of motion, it is impossible to formulate and to solve the problem in the previously intended representation since an additional fluent is missing. Nevertheless, a new problem had been established by the reduction of the motion in a plane to a motion along a straight line. Then, the path is never a curved line and the problem has to be solved how to find a relation between

[36] Newton argued that there is in the *real world* always a dissipation of motion, i.e. a transformation which does not result in a conservation of the same quantity, but in a loss of motion. Nevertheless, this argument cannot be justified for a model where the world consists only of a limited number of bodies (compare Sect. 4.4).

[37] Hence, from the mechanical interpretation of the assumption made by Newton to formulate the two problems to be solved ("Calculate the relation (ratio) between fluxions, if the relation (ratio) between fluents are given", compare Chap. 3 above), it follows that Newton implicitly assumed the validity of a general relation between the fluents and fluxions of path and velocity which is equivalent to the energy conservation of motion in the presence of conservative forces.

the fluent and its fluxions. Reformulating Eq. (3.15), the *fluent x* and the *fluxion of the fluxion* $\ddot{x}$ are mutually related to each other.

$$k \cdot x + \dot{p} = 0 \quad \text{or} \quad k \cdot x + m \cdot \ddot{x} = 0 \tag{3.16}$$

Newton's intention was to calculate the relation between the fluxions, if the relation between the fluents is given. The modification of the previous question is due to the reduction of motion from *two* to *one* dimension. Then, there is only one fluent and one fluxion instead of two correlated fluents and two correlated fluxions.

3.3 Leibniz's Foundation of the Calculus

Leibniz's foundation of the calculus is also as Newton's theory of fluxion connected to two problems, first it originates from the same question to demonstrate the relation between curves and areas or between curves and tangents in a general form, and, second, it is continued to relate the geometrical curve, tangents and areas to the paths of bodies, i.e. the motion of bodies. Therefore, the relations between geometry and mechanics have to be defined in a new frame which includes the relation between geometrical figures of different type. Generally, the geometrical objects are classified into point, lines (straight or curved), areas and solids (filled regions of space[38]).

Then, if follows that

(i) the foundation is related to the fluents and fluxions where the time is implicitly or explicitly included and which is closely related to the motion of bodies,
(ii) the foundation is related to points, tangents, curves and areas which are not implicitly or explicitly related to time and the motion of bodies,
(iii) the motion of bodies is the starting point for the invention of the calculus as far as the very beginning of motion is considered [Newton, Principia], [Leibniz, Specimen] which is distinguished from motion related to finite distances in finite time.

From (iii) it follows that one has to describe these two types of motion differently, using different mathematical languages which correspond to the distinction between

[38] The plenum is to be distinguished from the empty space. For mechanics, Descartes assumed the *plenum*, the space without empty regions, whereas in *geometry*, the solidum is defined as region which is empty and which can be limited in extension by bodies. The solidum is generated by the motion of a line creating a line, the subsequent motion of a line creating a plane, followed by the subsequent motion of a plane creating a solidum [Leibniz, Initia]. This model has been also used by Newton [Newton, Principia]. The origin can be found in ancient time ([Heron] cited by Tropfke [Tropfke]). Obviously, the inverse problem is the creation of a geometrical point starting with the plenum. Leibniz claimed that the continuum (representing the plenum) does not consist of points [Leibniz, Theodizee, Preface].

conatus or *nisus* and *impetus*. Following Leibniz, the nisus is considered as *elementary*, i.e. as Leibniz explained, *infinitesimal*, which is also called solicitation.

> Hinc patet duplicem esse *Nisum*, nempe elementaremm seu infinite parvum, quem et *solicitationem* appello, et formatum continuatione seu repetitione Nisuum elementarium, id est *impetum* ipsum. [Leibniz, Specimen, I (5)]

Here, we find the model of the generation of finite quantities, e.g. the Cartesian impetus or momentum, by infinitesimal or elementary quantities without changing the thing which is set in motion and which is finally moving. Mechanically, the invariant property is given by the *mass* which is independent of the different stages of motion. However, Leibniz claimed that the invariance is additionally explained by the inherent forces of the body, the *active* force and the *passive* force. Both forces are specified to be either *primitive* or *derivative*. Hence, there are *primitive active* and *derivative active* as well as *primitive active* as well as *derivative passive* forces [Leibniz, Specimen, I (3) and (4)]. The *derivative* forces are generated by the bodies which are mutual acting upon each other in their interaction. Moreover, Leibniz distinguished between *dead* and *living* forces [Leibniz, Specimen, I (6)]. Leibniz's model of different forces is crucial for the interpretation of the calculus since Leibniz established a model where an infinite number of *infinitesimal* impressions of dead forces result in a *finite* living force [Leibniz, Specimen I (8)].

In contrast to Newton, Leibniz preferred a foundation which is independent of time, but related to the transformation of forces of different type into each other. Moreover, Leibniz included those essential relations between *forces* which had been invented by Newton before in the *Principia* like the activation of the force of inertia by the impressed moving force.[39] Nevertheless, the basic rules invented by Leibniz are independent of a geometric or mechanical interpretation. The advantage of this approach is readily demonstrated since the infinitesimal quantities are related to each other by rules which follow solely from their specific mathematical properties. From the Leibnizian approach it follows that the foundation of the calculus should be related to "ordinary algebra".

Leibniz claimed that the "knowledge of calculus" makes it possible to "shed a new light upon geometry" and achieve things "beyond the province of that science".

> But Huygens, who as a matter of fact had some knowledge of the method of fluxions as far as they are known and used, had the fairness to acknowledge that a new light was shed upon geometry by this calculus, and that knowledge of things beyond the province of that science was wonderfully advanced by its use. [Leibniz, Historia]

Leibniz discussed the foundation and application of the calculus in the correspondence with Varignon, Jakob and Johann Bernoulli and his response to Nieuwentijt. In 1714, Leibniz summarized the development since the early attempts to invent the

[39] "Ex dictis illud quoque mirabile sequitur quod omnis corporis passio sit spontanea seu oriatur a vi interna, licet occasione externi." [Leibniz, Specimen, II (5)] "Exercet vero corpus hanc vim solum modo in mutatione status sui per vim aliam in se impressam facta; estque exercitium ejus sub diverso respectu & resistentia & impetus." [Newton, Principia, Definitions]

method in the paper *Historia et Origo calculi differentials* which was only published by Gerhardt in the 19th century [Leibniz, Historia].

Leibniz based his argumentation on a combined approach consisting of geometrical and arithmetical arguments. The relations between geometry based approach and algebra based approach had been analyzed by Cajori [Cajori], Brumbaugh [Brumbaugh] and others.

> As regards signs, I see it clearly that it is to the interest of the Republic of Letters and especially of students, that learned men should reach agreement on signs. Accordingly I wish to get your opinion, whether you approve of marking by the sign $\int$ the sum, just as the sign d is displayed for differences; (...) Perhaps it will be well to examine other symbols, concerning which more on another occasion. [Leibniz, Cajori, p. 182]

Leibniz considered differences and sums of higher order without a limitation in the order. However, assuming the application of the Leibnizian scheme to expressions composed of variables and constants, for any polynomial there is only a *final* sequence of differences whereas the sequence of sums is not limited by this procedure. In the simplest case of natural numbers, given by Leibniz, it is easily demonstrated that the differences of 2nd degree vanish (Table 3.1).

A new basis for the modelling and solution of the problem arose from the application of the calculus to mechanics [Varignon 1700], [Varignon, Infiniment], i.e. the motion of bodies where the finite and infinitesimal quantities are interpreted not only *geometrically*, i.e. related to distances, but also *temporally*, i.e. to bodies travelling a certain distance in a certain time. Although the problem becomes more complicated due to invention of quantities of different type like distances and time intervals the advantage of the procedures becomes evident since some of the previously disturbing ambiguities could now be removed, i.e. for the first step as far as the application to mechanics was concerned.

Euler introduced an alternative approach where the relation to geometry is not assumed. Therefore, the questions previously appeared as far as geometrical curves were analyzed do not appear. Euler based the calculus on the transfer of the arithmetical rules, i.e. addition, subtraction, multiplication and division, valid for *finite* quantities to the operation with infinitesimal quantities. As a consequence, Euler considered infinitesimal quantities of different magnitude which are related to infinite quantities of different magnitude [Euler E387, § 84]. All these quantities are considered as *numbers* [Euler E101, §§ 1 to 10] and are allowed to appear in expressions composed of infinitesimal, finite and infinite quantities. Two quantities are related to each other by mathematical reason to compare quantities of the same degree in the hierarchy of differences. Euler's approach can be traced back to Leibniz's procedure [Leibniz, Elementa].

Table 3.1 Leibniz's scheme of differences

Series	1,		2,		3,		4,		5, ...
Diff. 1st degree		1,		1,		1,		1, ...	
Diff. 2nd degree			0,		0,		0,		0, ...

Leibniz considered a basic set of number, their differences and sums [Leibniz, Historia]. Subtraction and addition are related to each other as inverse mathematical operations. There is no internal reason to compare the differences of second degree and the sums of second degree. Nieuwentijt considered only differentials allowed to be developed up to the second degree, the differentio-differentials [Nieuwentijt, Analysis].

The scheme presented in Table 3.2. demonstrates the advantage of the arithmetical approach which is completely independent of any mechanical or geometrical interpretation. The model of polygon and circle was appropriate to discuss the relation between straight and curved lines as well the relation between finite and infinite quantities. However, geometrically, the step from 2D to 3D, 4D and higher dimensions is a cumbersome procedure whereas it is straightforwardly performed by analytical methods [Euler E289] (compare Chap. 4). Nevertheless, also from the polygon and circle model it follows that different types of numbers have to be introduced and used *simultaneously*. As a consequence, Leibniz assumed implicitly that the rules for addition, multiplication and division assumed for finite numbers remain to be valid also in the case where the finite numbers are replaced with infinitesimal and infinite numbers. Additionally, the introduction of infinitesimal and infinite numbers cannot be performed independently of each other.[40] The only difference between Leibniz's procedure and the procedure later introduce by Euler [Euler E212] is the difference between the *implicit* and *explicit* formulation of the mathematical rules. Euler used to give all supposition he made in an explicit and comprehensible representation [Euler E015/016], [Euler E212], [Euler E387].

Table 3.2 Leibniz's interpretation of the calculus in terms of differences and sums

Diffs.		1		2		3		4		5		. . . .		dx
Series	0		1		3		6		10		15	. . . .		x
Sums		0		1		4		10		20		25	. .	$\int x$

Leibniz introduced the calculus in procedures which are related to figures and to numbers and their difference. The third procedure discussed in the previous chapter is related simultaneously to figures represented by the paths of moving bodies and the measurement of the finite velocities of these bodies. The relations between the geometric and the arithmetic approach can be studied in the French edition *quadrature arithmétique du cercle, de l'ellipse et de l'hyperbole* of text book on Leibniz's approach recently edited by Knobloch [Leibniz, quadrature arithmétique].

[40] This procedure had been analyzed by Euler and later, in 20th century, by Laugwitz and Schmieden [Laugwitz Schmieden] and Robinson [Robinson], [Keisler].

3.3.1 Nova Methodus

The calculus was invented by Newton as the formalism for the relations between time related quantities called fluents and fluxions. Leibniz preferred a foundation which is independent of time (compare comment on Cavalieri and Fabri). In 1684, Leibniz published a paper entitled *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irationales quantitates moratur, et singulare pro illis calculi genus*[41] in Acta Eruditorum. This was for the first time the scientific community became familiar with the differentials introduced by Leibniz and, furthermore was prepared for an alternative version Newton presented some years later in 1687 [Newton, Principia]. Hence, the reader had the chance to compare both representations whose subject is, following Newton, one and the same thing [Newton, Principia, (Scholion on Leibniz)]. However, nobody did it, but the community split in two parts comprising those who made use of Leibniz's notation and algorithm and those who made use of Newton's notation and reckoning principles presented in the *Principia*.[42] Comparing the title of this paper to that one of the 1680 paper, the words "pro differentiis and summis" are missing in the later version.

The advantage of this approach is readily demonstrated since the infinitesimal quantities are related to each other by rules which follow solely from their specific mathematical properties. These properties are only related to the rules of other parts of mathematics, e.g. geometry and algebra. Although the calculus arises also from the needs of mechanics these mathematical rules are completely independent of an interpretation within a mechanical model.

The basic rules of the calculus are valid for quantities x and y which are either constants or can be *increased* or *decrease* by adding or subtracting finite or infinitesimal quantities. For increase and decrease of infinitesimal quantities the validity of the same rules are assumed which are valid for finite quantities [Leibniz, Nova Methodus]. In case of addition and subtraction, Leibniz assumed that the arithmetical rules of addition and subtraction valid for finite quantities z,y,w,x is transferred to the same arithmetical operations for differentials dz,dy,dw,dx.

Leibniz assumed that finite quantities are added according to the same rules as infinitesimal quantities had to be added and vice versa. A modification appears in the case of *multiplication* and *division*. Leibniz's basic assumption had been already discussed: "Also $da = 0$, if it is given that a is a *constant* quantity, since $a - a = 0$." [Leibniz, Elementa]. In 1684, Leibniz started with the theorem on the existence of an *assignable* differential. The differential $da = 0$ is not defined as a magnitude less than any *assignable* quantity since zero is an *assignable* quantity, but as a magnitude

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[41] An earlier version is entitled "Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum, solidorum, aliisque communem calculum transcendentibus." "The elements of the new calculus for differences and sums, tangents and quadratures, maxima and minima, dimensions of lines, surfaces, and solids, and for other things that transcend other means of calculation." [Leibniz, Elementa] In 1684, the words "differentiis et summis" are missing.

[42] The next opportunity Newton gave to the community to understand his approach was only 20 years later when the *Quadrature* had been publish in 1704 [Newton, Opticks].

less than any finite quantity and equal to zero. Leibniz commented on the status of differentials.[43]

> Sit a quantitas data constans, erit da aequalis 0, et $d\overline{ax}$ erit aequalis adx. Si sit y aequ. v (seu ordinata quaevis curvae YY aequalis cuivis ordinatae respondenti curvae VV) erit dy aequ. dv. Jam *Additio et Subtractio*: si sit $z-y+w+x$ aequ. v, erit $d\overline{z-y+w+x}$ seu dv aequ. $dz-dy+dw+dx$. *Multiplicatio*: $d\overline{xv}$ aequ. $xdv+vdx$, seu posito y aequ. xv, fiet dy aequ. $xdv+vdx$. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notamdum, et x et dx eodem modo inhoc calculo tractari, ut y et dy, vel aliam literam indeterminatam cum sua differentiali.[44] Notamdum etiam, non dari semper regressum a differentiali Aequatione,[45] nisi cum quadam cautione, de quo alibi. Porro *Divisio*: $d\frac{v}{y}$ vel (posito z aequ. $\frac{v}{y}$) dz aequ. $\frac{\pm vdy \mp ydv}{yy}$. [Leibniz GM V, p. 220]

The basic rule of the calculus is valid for quantities x and y which are either constants or can be *increased* or *decrease* by adding or subtracting finite or infinitesimal quantities. For increase and decrease by infinitesimal quantities the validity of the same algebraic rules are assumed which are valid for finite quantities [Leibniz, Nova Methodus].

$$d\overline{z+y-w+x} = dz+dy-dw+dx. \tag{3.17}$$

A new rule is introduced for the operation applied to a product of two quantities which are assumed to be of finite magnitude

$$d\overline{xy} = ydx+xdy \tag{3.18}$$

is independent of the interpretation of the variables x and y as *time-dependent* or *time-independent* coordinates or other time-dependent or time-independent variables, e.g. the velocity.[46] From these rules it follows

$$d\overline{xx} = 2xdx \quad \text{and} \quad d\overline{x^n} = nx^{n-1}dx \tag{3.19}$$

which may by generalized for negative powers

$$d\overline{x^{-1}} = -x^{-2}dx \tag{3.20}$$

$$d\overline{x^{-n}} = -n\cdot x^{-n-1}dx. \tag{3.21}$$

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[43] "Der Beweis alles dessen (nämlich der Differentiationsregeln) wird für einen in diesen Erfahrenen leicht sein, wenn er den bisher nicht genug erwogenen Umstand beachtet, dass man die dx, dy, dv, dw, dz als proportional zu den augenblicklichen Differenzen; d.h. Inkrementen oder Dekrementen der x, y, v, w, z ... betrachten kann." [Leibniz, Nova methodus] Obviously, Leibniz referred to Newton's model (compare the comment by Kowalewski [Newton, Quadratur (Kowalewski)]).

[44] The differential is indeterminate.

[45] It is not always possible to establish an equation for the differentials.

[46] Moreover, it would be very dangerous and inconvenient to base the mathematical operation on a physical background. From the very beginning, the critics of the calculus, Nieuwentijt [Nieuwentijt, Analysis] and later Berkeley [Berkeley, Analyst], attacked preferentially the differentials of higher degrees, especially the possibility to generate differentials of unlimited degree. Nieuwentijt accepted only the *differentials* and the *differentio-differentials* as reasonable quantities. Mechanically, these quantities are mainly related to velocity and acceleration.

Obviously, the latter relation may be hardly demonstrated geometrically, but follows straightforwardly from the algebraic rules.[47]

Defining the operation $d\overline{xv} = xdv + vdx$, Leibniz assumed that the expression $d\overline{xv}$ can be traced back to another expression being the difference of two products, $d\overline{xv} = (x+dx)(v+dv) - xv$. Here, the change of one quantity x is independent of the change of the other quantity y. It is only assumed that both the quantities are modified simultaneously, since for non simultaneous change it follows $da\bar{x} = a(x+dx) - ax = adx$. Therefore, in the complementary case, $dx\bar{a} = x(a+da) - ax = xda$. Therefore, adding these two relations, it follows $d\overline{ax} = da\bar{x} + dx\bar{a} = adx + xda$ and the obtained rule can be interpreted in terms of the following algorithm: The infinitesimal change of a product of two variable quantities is obtained as a sum of two terms where it is assumed that in each of the terms one constituent is kept constant, i.e. its magnitude is assumed to be invariant or independent of the change of the other constituent, whereas the other is changed. Then, it follows

$$d\overline{ax} = d(a_{inv}\bar{x}) + d(x_{inv}\bar{a}) = adx + xda. \tag{3.22}$$

This relation can be interpreted geometrically if we assume that an rectangular area is defined by the product $F = ax$. Geometrically, the Leibnizian operation means that one has to consider the infinitesimal areas which are obtained by shifting of the side 1 of length a by the infinitesimal distance dx and by the shifting of the side 2 of the length x by the infinitesimal translation da. The lengths of both sides are *not changed* by this procedure. Another result is obtained if the sides are not considered as invariant: $d\overline{xv} = (x+dx)dv + (v+dv)dx = xdv + vdx + 2dxdv$. Therefore, the Leibnizian procedure may be called the "partial variation" of the quantities. The differential for the ratio is obtained from the rule

$$d\overline{x^n} = nx^{n-1}dx. \tag{3.23}$$

generalized for negative numbers: $n = -1, -2, -3, \ldots$. The case of a constant my also interpreted geometrically since there is no change of the basic area. It is very hard to derive the rules for negative powers geometrically because of the negative sign which appears necessarily from the generalization of the rules for positive integers.

> It is demonstrated by the calculus as follows: dxy is the same thing as the difference between two successive (neighboured) $xy's$; let one of these be xy and the other $x+dx$ into $y+dy$; then we have
>
> $$d\overline{xy} = \overline{\overline{x+dx} \cdot \overline{y+dy}} - xy = xdy + ydx + dxdy;$$
>
> the omission of the quantity $dxdy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small (because the lines are understood to

[47] Leibniz could not make use of his relational definition of time and space for the solution of this problem since the *order of successions* is not appropriate to define a relation between the time intervals [Euler E149]. A similar objection was made by Clarke [Leibniz, Clarke, 3rd Letter to Leibniz]. Euler demonstrated that equal time intervals and equal space intervals are defined by uniform motion of a body.

> be continuously increasing or decreasing by very small increments throughout the series of terms), will leave $xdy + ydx$; the signs vary according as y and x increases together, or one increases as the other decreases; this point must be noted. [Leibniz, Elementa, (Child, p. 143)]

Contrary to the model of differences and sums, Leibniz interpreted the generation of lines not be the apposition of parts, but a by modification by very small increments throughout the series of terms or "cum scilicet per seriei terminum lineae continue per minima crescentes vel decrescentes intelliguntur" or "because the *ends of the lines* lines are understood to be continuously increasing or decreasing by very small increments throughout the series of terms".

In a letter to Johann Bernoulli, Leibniz summarized:

> I recognize (...) that you have written some profound and ingenious things concerning various infinite bodies [de corporibus varie infinitis]. I think that I understand your meaning, and I have often thought about these things, but have not yet dared to pronounce upon them. For perhaps the infinite, such as we conceive it, and the infinitely small, are imaginary, and yet apt for determining real things [but suitable for setting bounds to real], just as imaginary roots are customarily supposed to be. These things are among the ideal reasons by which, as it were, things are ruled, although they are not in the parts of matter. For if we admit real lines infinitely small, it follows also that lines are to be admitted which are terminated at either end, but which nevertheless are to our ordinary lines, as an infinite to a finite. Which things being posited, it follows that there is a point in space which can not be reached in an assignable time by uniform motion. And it will similarly be required to conceive a time terminated on both sides, which nevertheless is infinite, and even that there can be given a certain kind of eternity (as I may express myself) which is terminated. Or further that something can live so as not to die in any assignable number of years, and nevertheless die at some time. All which things I dare not admit, unless I am compelled by indubitable demonstrations. [Leibniz, GM III, pp. 499–500][48]

The infinite and the absolute do not consist of parts.

> Porro, ut nego rationem, cujus terminus sit quantitas nihilo minor, esse realem, ita etiam nego, proprie dari numerum infinitum vel infinite parvum, etsi Euclides saepe, sed sano sensu, de linea infinita loquatur. *Infinitum continuum* vel *discretum* proprie nec unum, nec totum, nec quantum est, (...). [Leibniz (1712) Acta Erudit] or [Leibniz GM V, 389]

This passage is quoted from Leibniz's paper *Observatio quod rationes sive proportiones non habeant locum circa quantities nihilo minors, et de vero sensu methodi infinitesimales* [Leibniz (1712) Act Erudit], [Leibniz GM V, 387]. Leibniz proposed an interpretation of infinitesimals by a comparison of bodies of different extension (see next Section).

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[48] "(...) Sunt ista in rationibus idealibus, quibus velut legibus res reguntur, etsi in materiae partibus non sint. (...) quo posito, sequitur esse punctum in spatio, ad quod hinc nullo unquam tempore assignabili per motum aequabilem perveniri possit; (...). Reale infinitum fortasse est ipsum absolutum, quod non ex partibus conflatur, sed partes habentia, eminenti ratione et velut gradu perfectionis comprehendit." [Leibniz, GM III, pp. 499–500]

3.3.2 Leibniz's Comments on the Calculus

In 1673–1678 Leibniz invented the calculus based on geometry and arithmetic. These papers had been only published in the 19th century [Gerhardt, Leibniz], [Pertz, Leibniz (Pertz)], [Child]. Although the invention of the calculus was superimposed by the examination of other problems, the main principles including the interpretation of differentials as fictitious quantities had been developed in that early period.[49] Later in 1680 and 1684, Leibniz separately examined the calculus without taking direct recourse to the former investigations of the continuum and the problem of motion.[50] In the treatise entitled *Elementa calculi novi* we have the complete description of the method which is (i) based on arithmetic (differences and sum), (ii) applied to curves (tangents and quadratures), (iii) geometry (lines, surfaces and solids) and (iv) other subjects which could not be treated with the commonly known methods. The only missing topic is the application to mechanics or any relation to the temporal dependence of the generation of curves or other geometrical figures by motion. The latter model had only invented by Newton, but it was known to be developed by the Ancients [Tropfke].

In 1705, Leibniz referred Newton's Quadrature. In 1712, Leibniz commented on the infinite. Following Leibniz, the infinite appeared in two forms as the (i) *continuous* infinite and (ii) the *discrete* infinite. The status of the differentials is closely related to the status of the infinite. Leibniz characterized the differentials and the infinite using the same word to be a "modus loquendi" whose application is always justified when two or more than two things are compared to each other like the orbits of fixed stars, the diameter of the earth and the grain of dust. Here, although Leibniz intended to characterize mathematical quantities, he established a model consisting in essential features of non-mathematical components. As a consequence, there is no clear and consistent distinction between continua of different kind related to (i) geometry and to (ii) mechanics. In mechanics, time and space are defined as order of successions and coexistences of *discrete* things by Leibniz [Leibniz, Initia] whereas Newton defined time and motion by *continuous* fluxes [Newton, Quadrature] (compare Chap. 2). Leibniz did neither consequently argue mathematically or arithmetically nor consequently geometrically, phenomenologically and mechanically.

> Sunt tamen quidem, ut sic dicam, *tolerabilitatis*. Porro, ut nego rationem, cujus terminus sit quantitas nihilo minor, esse realem, ita etiam nego, proprie dari numerum infinitum

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[49] Recently, Arthur analyzed this development of Leibniz's foundations and interpretations of the calculus. "Leibniz's brilliant mathematical innovations are in constant interplay with his thoughts on natural philosophy and its metaphysical foundation." [Arthur, Fictions], [Leibniz, Continuum] Later in the 19th century, this interplay disappeared and remained only rudimentary although the progress was mainly due to the recreation of the previous procedure as it can be demonstrated for the development of the theory of relativity and quantum mechanics.

[50] In 1686, 1695 and 1698 similarly to the previous procedure in examining the calculus, Leibniz separately investigated the basic principles of *mechanics* without taking direct recourse to the foundation of the calculus and the results he had meanwhile obtained [Leibniz, Brevis], [Leibniz, Specimen], [Leibniz, De ipsa], [Leibniz, Phoranomus]. On the contrary, Newton made use of the method of fluxion, but the application and the foundation of the calculus was only fragmentarily presented to the public [Newton, Principia].

> vel infinite parvum, etsi Euclides saepe, sed sano sensu, de linea infinita loquatur. *Infinituum continuum* vel *discretum* proprie nec unum, nec totum, nec quantum est, et si analogia quaedam pro tali a nobis adhibeatur, ut verbo dicam, est modus loquendi; cum scilicet plura adsunt, quam ullo numero comprehendi possunt, numerum tamen illis rebus attribuemus analogice, quem infinitum appellamus. Itaque jam olim judicavi, cum infinite parvum esse errorem dicimus, intelligi dato quovis minorem, revera nullum; et cum ordinarium, et infinitum, et infinities infinitum conferimus, perinde esse ac si conferremus ascendendo diametrum pulvisculi, diametrum terrae, et diametrum orbis fixarum, aut his quantumvis (per gradus) majora minoraque, eodemque sensu descendendo diametrum orbis fixarum, diametrum terrae, et diametrum pulvisculi posse comparari ordinario, infinite parvo, et infinities infinite parvo, sed ita ut quodvis horum in sua genere quantumvis majus aut minus concipi posse intelligatur. [Leibniz, 1712, Acta Erud. GM V, 389][51] Jesseph commented.[52]

The correlation between mathematics and physics is as impressive as possible, but suffers from the disadvantage that space and time are not included into the consideration by the same procedure. Neither a grain of dust nor the diameter of the earth nor the sphere of the fixed stars can be used to model a relation between purely mathematical or dimensionless numbers. However, there is also an interpretation which only refers to the rules Leibniz had been given in 1680 and 1684. In 1680, Leibniz introduced differentiation and integration as operations inverse to each other, (i) $\int dx = x$ (ii) $d\int x = x$. From these assumptions, it follows that there are *differentials* to whom a real number can be assigned if the operation $\int dx = x$ is accepted to be valid: the finite quantity x is assigned to the differential dx. Furthermore, the *real number zero* is assigned to the differential of a constant quantity $a = \text{const}$ whereas the differential dx of a variable $x \neq 0$ is necessarily also different from zero. Although Leibniz defined the differential dx as an evanescent difference of two finite quantities whose values approached to each other, the difference between the

[51] "(...) infinite parvo, sed ita ut quodvis horum in sua genere quantumvis majus aut minus concipi posse intelligatur. Cum vero saltu ad ultimum facto ipsum infinitum aut infinite parvum dicimus, commoditati expressionis seu breviloquio mentali inservimus, sed non nisi *toleranter vera* loquimur, quae explicatione *rigidantur*. Atque haec etiam mea sententia est de areis illis Hyperboliformium Asymptoticis, quae infinitae, infinitiesque infinitae esse dicuntur, id est talia rigorose loquendo vera non esse posse, tamen sano aliquo sensu tolerati. Atque haec tum ad terminandas virorum clarissimorum Varignonii et Grandii controversias, tum ad praecavendos chimericos quosdam conceptus, tum denique ad elidendas oppositiones contra methodum *infinite similem* prodesse possunt." [Leibniz, 1712, Acta Erud. GM V, 389]

[52] "Likewise, the fictionality of the infinitesimals is state in language that seems to have been almost borrowed from Hobbes. In objecting to the notion that there could be a proper ratio between positive and negative quantities, Leibniz remarked: 'just I have denied of the reality of a ratio, one of whose terms is less than zero, I equally deny that there is properly speaking an infinite number, or an infinitely small number, or any infinite line or any line infinitely small. ... The infinite, whether continuous or discrete, is not properly a unity, nor a whole, nor a quantity, and when by analogy we use it in this sense, it is a certain *façon de parler*, I should say that when a multiplicity of objects exceeds any number, we nevertheless attribute to them by analogy a number, and we call it infinite. And thus I once established that when we call an error infinitely small, we wish only to say an error less than any given, and thus nothing in reality. And when we compare an ordinary term, an infinite term, and one infinitely infinite, it is exactly as if we compare, in increasing order, the diameter of a grain of dust, the diameter of the earth, and that of the sphere of the fixed stars." [Leibniz, 1712, Acta Erud. GM V, 389] Jesseph [http://www4.ncsu.edu/~dmjphi/Main/Papers/Truth%20in%20Fiction.pdf]

assignable and *unassignable* quantities is not removed, but is still in power. The difference is traced back to the distinction between assignable variables and assignable constants. Choosing a certain value of the variable $x = x_0$, the differential becomes of assignable magnitude $dx_0 = 0$. Leibniz' assumptions can be summarized as follows:

> Also $da = 0$, if it is given that a is a *constant* quantity, since $a - a = 0$. [Leibniz, Elementa, 1680]

Consequently, following Leibniz, the differential of a constant is a *determinate* and an *assignable* quantity whereas the differential of a variable is an *indeterminate* and *unassignable* quantity. Completing Leibniz's theorem by the explicit rule which had been given only implicitly, it follows:

> Also $dx \neq 0$, if it is given that x is *not a constant* quantity

Straightforwardly, any indeterminate quantity *different from zero* can take either the positive or the negative sign or can be analytically represented by $-dx < 0 < +dx$ (I). Following Euler and generalizing the *principle of continuity* to discrete quantities, the inequality can be completed by the inclusion of *all finite quantities* with, as before, the only exception of the number zero or $-x < 0 < +x$ (II). Combining (I) and (II) we obtain the relation

$$-x < -dx < 0 < +dx < +x \tag{3.24}$$

where the well-known order between positive and negative numbers is established.

$$\ldots . - x, -dx, 0, +dx, +x, \ldots .. \tag{3.25}$$

It can be readily seen that the crucial point is Euler's generalization of the principle of continuity to series of discrete terms [Euler E212, § 85]. In 1716, Leibniz reinforced the interpretation of differentials as fictitious quantities [Leibniz, Letter to Dangicourt]. Nevertheless, Leibniz paved the way for an arithmetical foundation beside the geometrical foundation of the calculus. From the further development, it had been confirmed that it is much easier to demonstrate the theorems of the calculus analytically instead of geometrically. The application to geometry became quite natural having analytically demonstrated all theorems in advance.[53]

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[53] Not surprisingly, Newton was fully aware of the difficulties to describe the new concept geometrically, but refused to abandon geometry because of the inherent paradigmatic power of geometry. Finally, it took two centuries to complete the procedure initiated by Leibniz and Newton, continued by Taylor, Maclaurin, Euler, d'Alembert, Lagrange, Cauchy, Dedekind and Weierstraß [Klein, Arithmetization]. "Perhaps it may be objected, that there is no ultimate proportion, of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument, it may be alledged, that a body arriving at a certain place, and there stopping, has no ultimate velocity: because the velocity, before the body comes to the place, is not its ultimate, velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases.

3.4 The Calculus: Development, Criticism and Controversies

In 1684, Leibniz published a systematic, but only very short exposition of the calculus where he outlined all basic algorithms for the first time. However, it took some years for the first reactions, appreciations and also for the first criticism. Soon, Jakob and Johann Bernoulli appreciated the new method and proved it to be powerful by application to various problems. Only three years later in 1687, Leibniz received a lot of support given to him by the great Newton who commented on the method in a short note published in the *Principia* [Newton, Principia, Book I, Sect. I, Lemma II]. Newton indirectly stated that he is convinced of the importance and the advantage of the method since the same approach had been invented by him twenty years ago and successfully applied to mathematical and mechanical problems outlined in the *Principia*. Hence, the great Newton and the celebrated Leibniz developed the same algorithms which had been only presented in "different words". This beautiful appreciation and appraisal lasted only some years up to the mid of the 1890's where the first critics asked for a reliable foundation and, moreover, Leibniz was accused plagiarism by Fatio [Fatio] and Keill [Keill]. Neither Fatio nor Keill analysed and accentuated the *differences* in Newton's and Leibniz's foundation of the calculus, but preferentially emphasized the common background by accentuation the *difference* of both approach relatively to the commonly accepted methods of the *predecessors*. Nevertheless, the further development was dominated by these two questions, first (i) the differences observed not only in the foundation, but also in the representation of the calculus by Newton and Leibniz and, second (ii) the progress made by both authors in comparison to all of their predecessors. Not surprisingly, also the critics of the calculus who soon appeared in the same time [Nieuwentijt, Analysis] and later [Berkeley, Analyst] accentuated not the differences, but in first respect the common principles in Newton and Leibniz including the resulting discrepancies and inconsistencies in the treatment of infinitesimal quantities. However, none of the insightful critics went decisively beyond the frame which had been explicitly established by Newton and Leibniz who were also not satisfied with the state of art in foundation, but were seeking for answers to the critics [Leibniz, Responsio], [Hermann, Responsio].

> In it, however, there is something that is worthy of all praise, in that he desires that the differential calculus should be strengthened with demonstrations, so that it may satisfy the rigorists; (…). [Leibniz, Cum]

And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating any other thing that is likewise geometrical." [Newton, Principia, Book I, Sect. I, Scholion]

The main objection was directed against the status and the use of infinitesimal quantities. Like their critics, Newton and Leibniz also intended to get rid as much as possible of infinitesimal quantities and emphasized that all demonstrations can be given in terms of finite quantities. Neither Newton nor Leibniz made the implicit or hidden assumptions on the reckoning with infinitesimal quantities explicitly known which lies behind of infinitesimal quantities of quite different magnitude. Finally, these rules can be only justified by the decision to be ready for making supplementary heresy,[54] i.e. the introduction of infinite quantities of different magnitude. This decisive step had been only done by some of followers in the 18th century, first of all by Euler.

> 84. It may be necessary also, in this place, to correct the mistake of those who assert, that a number infinitely great is not susceptible to increase. This opinion is inconsistent with just principles we have laid down; for 1/0 signifying a number infinitely great, then 2/0 being incontestably the double of 1/0, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater.[55] [Euler E387, Algebra][56]

In the 18th century, Taylor [Taylor, Methodus] and Euler [Euler 1727], [Euler E212] invented an entirely arithmetical foundation of the calculus by the invention of the calculus of finite differences as the foundation of the differential calculus

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[54] Similarly, Leibniz's measure of living forces was as successful as only incompletely demonstrated to be valid for some decades until the 1730's years (compare Chaps. 4 and 7). "567. On a vu au chap. 13 qu'il est démontre par la théorie de Galilée que les espaces que la gravité fait parcourir aux corps qui tombent vers le terre, sont comme les quarrés des vitesses (...). Cette assertion parut d'abord une espèce d'Hérésie Physique. *D'où viendroit ce quarré*, disoit-on?" [Châtelet, Institutions, § 567] (see also [Reichenberger])

[55] Regrettably, Euler's denotation in this writing is misleading, but the demonstration can be confirmed by non-standard analysis if the "0" denoting an infinitesimal quantity is replaced with an appropriate notation of quantities of infinitesimal magnitude which had been already used by Euler in *Instituitiones* [Euler E212] (compare Chap. 5). Not surprisingly, almost the same consideration is found in Newton's unpublished papers. "Thus 2/0 is double to 1/0 & 0/1 is double to 0/2, for multiply the 2 first & divide the 2nd by 0, & there results (2/1):(1/1)&(1/1):(1/2)(...). This is indefinite (that is undetermined) how great a sphere may be how great a number may be reckoned, how far matter is divisible, how much time or extension we can fansy but all the Extension that is, Eternity, $a/0$ are infinite." [Krantz] Here, Newton made use of the rule which he has later applied to the calculation of first and last ratios, one infinitesimal quantity ε divided by another infinitesimal quantity of the same magnitude ε results in a finite quantity, or, the ratio $\varepsilon/\varepsilon = 1$ is equal to one. In 2003, Krantz commented: "It is safe to say that Isaac Newton never fully came to grips with the notion of infinitesimal, but he made peace with it well enough to develop the science that he wanted to develop. (...) It is difficult, from our modern Olympian perspective, to understand the mindset of Newton's days. Even the concept of 'velocity' was relatively new at that time. While Newton was inventing mechanics, he was also inventing the very *language* in which it is expressed. And Newton's great nemesis, in all his ruminations, was the concept of infinitesimal. He constantly had to confront Zeno's paradox, and the many apparent contradictions arising therefrom, in all his considerations of fluxions and fluent and quadrature and acceleration." [http://www.ams.org/notices/200311/rev-krantz.pdf]

[56] "84. Hier ist es nöthig, noch einen gewöhnlichen Irrthum aus dem Wege zu räumen, der in der Behauptung besteht, ein unendlich Großes könne weiter nicht vermehrt werden." [Euler E387, § 84] "No matter what kind of quantity it may be, we should understand that every quantity, no matter how large, can always be made greater and greater, and thus increased without limit, that is, increased to infinity." [Euler E212, § 72] (compare also [Euler E389, §§ 1–6] and Chap. 5)

(compare Chap. 5).[57] Hence, the "heresy" consists in the *complete substitution* of geometry with algebra and arithmetics. As it had been demonstrated above and will be reviewed in the next sections, Newton and Leibniz were only ready for a *partial* substitution and a development of geometry and algebra in parallel.[58] Following Klein, this development had been finally completed by Weierstraß [Klein, Arithmetization].[59] Nevertheless, Leibniz emphasized the role of algorithms.

> Ex cognito hoc velut *Algorithmo*, ut ita dicam, calculi hujus, quem voco *differentialem*, omnes aliae aequationes differentiales inveniri possunt per calculum communem, maximaeque et minimae, itemque tangentes haberi, ita ut opus non sit tolli fractas aut irrationales aut alia vincula, quod tamen faciendum fuit secundum Methodos hactenus editas. Demonstratio omnium facilis erit in his rebus versato et hoc unum hactenus non satis expensum consideranti, ipsa dx, dy, dv, dw, dz, ut ipsarum x, y, v, w, z (cujusque in sua serie) differentiis sive incrementis vel decrementis momentaneis proportionales haberi posse. [Leibniz, Nova methodus]

Following Leibniz, the quantities x, y, v, w, z are independent of each other and, consequently dx, dy, dv, dw are also not correlated. Following Newton, neither the quantities x, y, v, w, z nor the moments dx, dy, dv, dw are independent of each other, but are correlated by the equably flowing time. Hence, at first, Leibniz distinguished the representation of the sum by the letter v and the representation of the terms constituting the sum by letters z, y, w, x and, at second Leibniz transferred this procedure to the increment/decrement of the sum and the increments/decrements of the terms. Instead of the Newtonian "simultaneity" [Newton, Method of Fluxions] due to the equable and universal flow of time or $v(t) = z(t) + y(t) - w(t) + x(t)$, the simultaneous change of the variables is indicated by the common bar above the variables $d\overline{z+y-w+x}$. Then, without changing the arithmetical rules, the expressions for finite quantities

[57] Supposing Newton's definition of *genitum* [Newton, Principia, Book II, Sect. II, Lemma II] it follows that (i) *infinitesimal* quantities of *different* magnitude are generated by the *multiplication* with the "only infinitely little quantity o", i.e. $a \cdot o$ is different from $b \cdot o$, or, $a \cdot o \neq b \cdot o$ if a is different from b, or, $a \neq b$, whereas, by the same reason, (ii) *infinite* quantities of *different* magnitude are generated by the *division* by the "only infinitely little quantity o", i.e. a/o is different from b/o if a is different from b, or, $a \neq b$.

[58] The decisive step in separating both disciplines to made arithmetics unchained to geometry had been done by Euler. Making use of Leibniz's methodology, Euler transferred the Leibnizian *principle of continuity* to *series* made up of *discrete terms* formed by those quantities like positive and negative number including zero [Euler E212, §§ 85–92] (compare Chap. 5). Then, following Galileo (compare Chap. 1), the *continuity of motion* of a falling body is represented by the *discrete series of odd numbers* 1, 3, 5, 7, ... which are mechanically interpreted to describe the increments of the path of the moving bodies in equal times.

[59] "Ich möchte (...) meine Stellung zu derjenigen wichtigen mathematischen Richtung bezeichnen, als deren Hauptrepräsentant *Weierstraß* dasteht, (...) – zur *Arithmetisierung der Mathematik*. (...) Da zeigte sich bei näherer Untersuchung, dass ... die Raumanschauung dazu geführt hatte, in übereilter Weise Sätze als allgemeingültig anzusehen, die es nicht sind. Daher die Forderung nach *ausschließlich arithmetischer Beweisführung*. (...) Wo sonst Figuren als Beweismittel dienten, da sind es jetzt immer wiederholte Betrachtungen über Größen, die kleiner werden oder angenommen werden können, als jede noch so kleine vorgegebene Größe." [Klein, Arithematization]

$$v = z + y - w + x \tag{3.26}$$

$$K = az^2 + by^2 + ew^2 + fx^2 = \text{const} \tag{3.27}$$

are transferred into the correlated expressions for infinitesimal quantities or the increments of the finite quantities.

$$d\bar{v} = d\overline{z + y - w + x} = dz + dy - dw + dx \tag{3.28}$$

$$d\overline{K} = 2azdz + 2bydy + 2ewdw + 2fxdx = 0 \tag{3.29}$$

Leibniz established a correlation between the differentials of the variables which are related to each other by these relations.

Newton had to deal with the same problem. Newton assumed a simultaneous change of two spaces represented by the variables y and x where the analytical correlation is established by the equation $xx = y$ between the spaces described by the motion of a body.

> LII. (...) de sorte qu'à la fine il n'en reste plus pas un qui ne soit plus petit que la plus petite Quantité donnée, & infiniment petit ou égal a zero, si on suppose l'Opération continuée à l'infini; (...).
>
> LVI. 1. La longueur de l'Espace décrit étant continuellement donné, trouver la vitesse du Mouvement à un tems donnée quelconque.
>
> LVII. 2. La vitesse du Mouvement étant continuellement donnée, trouver la longueur de l'Espace décrit à un tems donné quelconque.
>
> LVIII. Ainsi dans l'Equation $xx = y$, si y représente la longueur de l'Espace décrit à un tems quelconque, lequel tems un autre Espace x en augmentant d'une vitesse uniforme $\dot{x}$ mésure & représente comme décrit, alors $2x\dot{x}$ représentera la vitesse avec laquelle dans le même instant l'Espace y viendra à être décrit & *vice versa*; & c'est de-là que j'ai dans ce qui suit consideré les Quantités comme produites par une augmentation continuelle à la maniere de l'Espace que décrit un corps en mouvement. [Newton, Method of Fluxions, LII, LVI, LVII and LVIII]

Obviously, there is an incompatibility between Newton's arithmetical and mechanical interpretation of paths and velocities. Newton assumed (i) a relation between fluents $x + y - z = 0$ which is mathematically equivalent to the relation $x + y - z = \text{const}$ as long as the constant had not been specified and (ii) the corresponding relation between fluxions $\dot{x} + \dot{y} - \dot{z} = 0$ and $\dot{x} + \dot{y} - \dot{z} = 0$, respectively [Newton, Method of Fluxions, LII, LVI, LVII, LVIII Problem I]. Mechanically interpreted, the relations $v = x + y - z = \text{const}$ and $\dot{v} = \dot{x} + \dot{y} - \dot{z} = 0$ describe a mechanical system where the quantity v is conserved whereas the "motion" is "redistributed" among the bodies whose paths are represented by x, y, z. The mechanical interpretation can be transferred to velocities u, v, w of moving bodies whose "living force" are conserved [Leibniz, Specimen, UV (11)]: $au^2 + bv^2 + ew^2 = \text{const}$. Then, it follows that the "motion" is redistributed among the interacting bodies: $2a\dot{u}u + 2b\dot{v}v + 2e\dot{w}w = 0$ where interaction is described by $\dot{u} \neq 0, \dot{v} \neq 0, \dot{w} \neq 0$.

Making use of Leibniz's notation and interpretation of differentials as *evanescent* quantities, it follows (i) $2audu + 2bvdv + 2ewdw = 0$ and (ii) $du \neq 0, dv \neq 0, dw \neq 0$ instead of Newton's $\dot{u} \neq 0, \dot{v} \neq 0, \dot{w} \neq 0$. The same result is obtained, but the interpretation is quite different. In Leibniz's frame, the differentials are interpreted as *fictitious* quantities and are not appropriate to describe *real changes* of motion.

In 1695, Nieuwentijt invented a model for differentials to truncate the hierarchy of infinitesimal quantities of different magnitude and to restrict the method to reckoning with differentials and differentio-differentenials [Nieuwentijt, Analysis].[60] The underlying idea and intention was not only to truncate the hierarchy of differentials, but also the hierarchy of infinite quantities being correlated with the former hierarchy. Nieuwentijt intended to eliminate the tendencies toward atheism, but to preserve the reckoning with differentials.[61]

In 1700, Varignon invented a geometry independent representation of mechanical quantities in terms of *differentials* and *differentio-differentials* centred upon the notion of uniform motion. The idea of uniform motion where a body is travelling *equal* distances in *equal* time intervals in one and the same direction can be readily interpreted as a *non-mathematical* and simultaneously a *non-geometrical* model for the calculus.

> It was by means of the geometry of infinitesimals that M. Vargnon reduced various motions to the same rule as uniform [motions], and it does not seem that he could have done so by any other method. [Fontenelle, Histoire]

The velocity can be considered either as a constant or as a variable quantity, represented by the relations $v = \text{const}$ and $v \neq \text{const}$, respectively. The third case is the state of rest or $v = 0$. Then, making use of Euler's terminology for the indication of infinitesimal quantities [Euler E212] (compare Chap. 5), the velocity $v = 0$ is not an infinitesimal quantity, but really nothing or zero, whereas the infinitesimal quantity dv is only "nothing or zero" in arithmetic ratios with respect to finite quantities,

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[60] Recently, Nieuwentijts approach had been reconsidered and compared to the analysis of Zeno's paradoxes by Verelst [Verelst].

[61] "Did Berkeley develop this criticism of the calculus independently from other critics? At the end of his essay 'The Analyst controversy' John Wisdom writes: '(...) it should be mentioned that prior to Berkeley a Dutchman put forward some criticisms of infinitesimals. There were thus two streams, perhaps partly parallel and partly intermixed' (*Hermathena*, no. LIV, 1939, p. 29). In fact, this Dutchman, Bernard Nieuwentijt (1654–1718), may even have provided Berkeley with the idea that there is a close connection between infinitesimal reasoning and atheism. In the *Considerationes* of 1694 Nieuwentijt criticized the first lemmas in Newton's *Principia* in which it is supposed that infinitesimals may be disregarded, because this supposition leads to absurdities (pp. 9–15). Moreover he questioned the validity of the procedure Newton follows to obtain the basic rules of differentiation (pp. 24–27). However, the bulk of Nieuwentijt's mathematical work is devoted to proving that the use of higher-order infinitesimals (such as $(dx)^2$, $dxdy$), and particularly the way in which Leibniz employs them, leads to contradictions and is dangerous from a religious point of view. Adoption of higher-order infinitesimals Nieuwentijt regarded as dangerous, in that it might lead to the assumption that man can grasp something of the infinite. This would obscure the fact that 'while it was our Creator's will that we are created in such a way that, although our comprehension can show us a quantity which is greater or smaller than whatever perceived quantity, we are still only able to perceive finite and determined objects; the human intellect is not capable of rising to a true and adequate understanding of the infinite itself' (*Analysis infinitorum* (1695), praefatio p. 4)." [Vermeulen]

$w \pm dv = w$ [Bernoulli 1691–1692], [Euler 1727], [Euler E212]. Hence in the frame of the mechanical model, the difference in magnitude between $v = 0$, or rest, and a differential of the velocity dv, or the change of rest, is a necessary component of the axiomatics [Newton, Principia, Axioms].[62] Euler categorized these different components as belonging either to *internal* or *external* principles [Euler E289] (compare Chap. 4).

In the end of the 19th century, Cantor referred to Leibniz. Cantor highly acknowledged Leibniz's description of differentials as fictitious quantities and accentuated, as Berkeley one hundred years before, the atheistic arnd theological implication of the interpretation of differentials as *determinate infinitesimal* quantities.

> So beruht z. B. die nicht selten vorkommende Auffassung der *Differentiale*, als wären sie *bestimmte* unendlich kleine Größen (während sie doch nur *veränderliche*, beliebig klein anzunehmende Hilfsgrößen sind, die aus den Endresultaten der Rechnungen gänzlich verschwinden und darum schon von *Leibniz* als bloße *Fiktionen* charakterisiert werden, z. B. in der Erdmannschen Ausgabe, S. 436), auf einer Verwechslung jener Begriffe. Wenn aber aus eine berechtigten Abneigung gegen solches *illegime* A.U. sich in breiten Schichten der Wissenschaft, unter dem Einflusse der modernen epikureistisch-materialistischen Zeitrichtung, eine gewisser *Horror Infiniti* ausgebildet hat, der in dem schon erwähnten Schreiben von Gauß seinen klassischen Ausdruck und Rückhalt gefunden, (...). [Cantor]

Nevertheless, Cantor paved the way for a new approach to infinity. His contemporaries were not ready to accept the new theory.

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[62] Mathematically interpreted, the state of rest is described by an infinitesimal quantity or the only real infinitesimal number (for the distinction between real and hyperreal numbers compare [Keisler] and Chap. 5). Euler accentuated this relationship and, by this reason, often represented the differentials by zeros [Euler E212]. However, the contemporaries and followers interpreted Euler's statements in a restrictive version that the differentials can be *always treated as zeros* although Euler insistently warned that one may fall into "maximal confusion" in doing so [Euler E212, § 85]. "85. Haec autem etiam in vulgari Arithematica sunt planissima: cuilibet enim notum est, cyphram per quemvis numerum multiplicatam dare cyphram, esseque n 0 = 0 [n cyph1 = cyph2], sicque fore n : 1 = 0 : 0 [n : 1 = cyph 2 : cyph 1]. Unde patet fieri posse, ut duae cyphrae quamcumque inter se rationem geometricam teneant, etiamsi, rem arithmetice spectando, earum ratio semper fit aequalis. Cum igitur inter cyphras ratio quaecumque intercedere possit, ad hanc diversitatem indicandam consulto varii characteres usurpantur; praesertim tum, cum ratio geometrica, quam cyphrae variae inter se tenent, est investiganda. In calculo autem infinite parvorum nil aliud agitur, nisi ut ratio geometrica inter varia infinite parva indagetur, quod negotium propterea, nisi diversis signis ad ea indicanda uteremur, in maxime confusionem illaberetur, neque ullo modo expediri posset." [Euler E212, § 85] "In der Infinitesimal=Rechnung aber thut man nichts anders, als daß man sich mit der Untersuchung des geometrischen Verhältnisses zwischen verschiedenen unendlich kleinen Größen beschäftigt, und dabey würde man in die größte Verwirrung gerathen ['aus der man sich nicht befreien könnte' missing passage which is not translated], wofern man nicht diese unendlich kleinen Größen mit verschiedenen Zeichen bezeichnete." [Euler E212, § 85 (Michelsen)] "In the calculus of the infinitely small, we deal precisely with geometric ratios of infinitely small quantities. For this reason, in the calculations, unless we use different symbols to represent these quantities, we will fall into the greatest confusion with no way to extricate ourselves." [Euler E212, § 85]

3.5 Berkeley

Berkeley's analysis concerns the suppositions, the reckoning procedure and the philosophical and theological implication which follow from the foundation and application of the calculus [Berkeley, Analyst]. The correlation of the mathematical foundation and the mechanical interpretation has also been accentuated by Berkeley who is afraid of the unpleasant consequences. Berkeley did not discuss the algorithms the calculus is based on, but the geometrical and mechanical interpretations. The confirmation is not obtained by the definition and the application of a self-consistent procedure which depends on help from other disciplines and is not able to make a business of its own.[63] The invention of the calculus has been comprehensively described by the creators Newton [Newton (Collins), Commercium], [Newton, Account] and Leibniz [Leibniz, Historia] in 1714 and 1715, respectively.

In the contemporary investigations in history of science, the different phases in the development and interpretation of the calculus are considered as being closely correlated with the other disciplines.[64] However, Berkeley did not accept that the algorithms reliable working are tools and the results of the application of the calculus are excellent since it was possible to formulate and to solve problems which could not be treated before with other methods.

Euler considered the calculus as based on the method of differences where only elementary mathematical operations like addition, subtraction, multiplication and division are applied for the calculation of the sums and difference of 1st, 2nd, 3rd and higher orders.[65]

> IV. By Moments we are not to understand finite Particles. These are said not to be Moments, but Quantities generated from Moments, which last are only the nascent Principles of finite Quantities. It is said, that the minutest Errors are not to be neglected in Mathematics: that the Fluxions are Celerities, not proportional to the finite Increments though ever so small; but only to the Moments or nascent Increments, whereof the Proportion alone, and not the Magnitude, is considered. And of the aforesaid Fluxions there be other Fluxions, which Fluxions of Fluxions are called second Fluxions. And the Fluxions of these second Fluxions are called third Fluxions: and so on, fourth, fifth, sixth, *&c. ad infinitum*. (...) And it seems still more difficult, to conceive the abstracted Velocities of such nascent imperfect Entities. But the Velocities of the Velocities, the second, third, fourth, and fifth Velocities,

[63] The need of such a procedure became obvious not only for the critics like Nieuwentijt (1695) and Berkeley (1734), but also for those who were involved in the development of mathematics and physics like the Bernoullis, Euler and d'Alembert. Later, the investigations were continued by Cauchy, Bolzano, Weierstraß and Cantor. Weierstraß invented a foundation which is completely independent of the confirmation by mechanical models.

[64] "Leibniz's brilliant mathematical innovations are in constant interplay with his thoughts on natural philosophy and its metaphysical foundation." [Arthur, Fictions] Later in the 19th century, this interplay disappeared or remained only rudimentarily although the progress was mainly due to the recreation of the previous procedure as it can be demonstrated for the development of the theory of relativity and quantum mechanics.

[65] This foundation is in goal and spirit a generalization of Leibniz's approach who called the differential calculus also the calculus of sums and differences [Leibniz, Historia]. Assuming this approach, the algorithms demanded by Johann Bernoulli are the same in the two main cases, first, for the calculus of finite and second for the calculus of infinitesimal differences [Euler E212].

> *&c*. exceed, if I mistake not, all Humane Understanding. (...) And if a second Fluxion be inconceivable, what are we to think of third, fourth, fifth Fluxions, and so onward without end? [Berkeley, Analyst]

However, Leibniz demonstrated that the same question already appear in case of the uniformly moving body along a straight line since the problem is does not consists in the mere construction of an appropriate explanation, but of an explanation which is in conformity to the established *principles of geometry* [Leibniz, Pacidius] or even better. Leibniz's idea is that one has to step beyond geometry in order to construct mechanics [Leibniz, Specimen]. Motion is different from geometry since a body is different from space (or vacuum) although space and bodies are extended things. However, Leibniz accepted this principle related to the difference between bodies and vacuum (empty space) only in the early period [Leibniz, Specimen, I (10)]. Later, Leibniz described the bodies by an ensemble of forces (compare Chap. 2) whose relations to the empty space are indeterminate. Hence, instead of constant, time dependent and coordinate dependent forces, Leibniz distinguished between (i) primitive and derivative, (ii) active and passive and (iii) forces represented by combinations of (i) and (ii), e.g. primitive active and primitive passive forces [Leibniz, Specimen, I (1)–(4)]. The most important distinction is introduced by the difference between *dead* and *living* forces [Leibniz, Brevis], [Leibniz, Specimen, I (6)].

Only reluctantly, Newton accepted the merits of Descartes [Westfall, Newton] and, being deeply convinced that the construction of the universe is based on sound geometry, he preserved a life-long prejudice against analytical methods introduced by Descartes [Westfall, Never]. Only recently, Newton's engagement in analyzing and commenting philosophical and theological subjects had been analyzed by Jesseph, Guicciardini, and other authors including the reformulation of mechanics in terms of the Leibnizian calculus [Guicciardini, Hermann], [Guicciardini, Reading].[66]

This complete mathematical and metaphysical including theological background had been only recently rediscovered but it was actually present and known in its full content for the followers of Newton and Leibniz in the 18th century. Moreover, the *mathematical* problems had not been treated separately from the *physical* and *metaphysical* background which were also always present, but in many cases not explicitly mentioned and accentuated. Even this relations had been analyzed and accentuated by Berkeley [Berkeley, Analyst] who argued

[66] "Newton's *Principia* remains one of the few undisputed masterpieces of mathematical physics that is still read. It is generally agreed to have established the foundations of dynamics and the theory of gravity, and to have given a remarkably accurate description of the motion of planets and their satellites. But there are few accounts of its reception, and what is usually said goes along these lines: the British loved it, but more discerning Continental mathematicians distrusted its physics and held out, initially and unsuccessfully, for a revival of the vortex theory of planetary motion which Newton had refuted. What really moved things on, however, was Euler's re-writing of this entire family of ideas in the language of the calculus, after which it was much easier for mathematicians to appreciate and extend what Newton had done. As this sketch indicates, a major part in the acceptance of Newton's ideas is supposed to have involved dumping his geometric analyses in favour of the systematic use of the calculus, which Newton had done so much to invent but largely left out of the *Principia*." [Guicciardini (Gray)]

against the cutting of the mathematical interpretation from other non-mathematical implications. Therefore, all followers were forced and stimulated to interpret the basic mathematical and mechanical concepts also metaphysically and theologically. This intimate connection became obvious in the criticism of the calculus by Nieuwentijt [Nieuwentijt, Analysis] and Berkeley [Berkeley, Analyst] who accused the authors of the calculus to promote infidelity. Therefore, it was an essential progress to direct and concentrate the dispute to the relation between a *geometrical* and *algebraic* (arithmetical or algorithmic) foundation. Ever since, geometry was considered as a well-established mathematical method accepted by all the authors. The invention of the *indivisibles* by Cavalieri and the *Epicurean* atomism by Gassendi caused trouble at first due to the atheistic implications and at second due to the questioning of the continuity as a fundament of geometry.

Chapter 4
Euler's Program for Mechanics

Die Naturlehre ist eine Wissenschaft, die Ursachen der Veränderungen, welche sich an den Körpern ereignen, zu ergründen.

Wo eine Veränderung vorgeht, da muß auch eine Ursache sein, welche dieselbe hervorbringt, weil gewiß ist, dass nichts ohne einen zureichenden Grund geschehen kann.

Die ganze Naturlehre besteht also darin, dass man bei einer jeglichen vorfallenden Veränderung zeige, in was für einem Zustand sich die Körper befunden, und dass wegen der Undurchdringlichkeit eben diejenige Veränderung habe entstehen müssen, welche wirklich vorgegangen.

Euler E842, §§ 1 and 50[1]

We shall now return to the main object of this paper – the discussion of the 'permanent' state of a system consisting of nuclei and bound electrons. For a system consisting of a nucleus and an electron rotating round it, this state is, according to the above, determined by the condition that the angular momentum of the electron round the nucleus is equal to $\hbar$.

Bohr, 1913

Very early, in 1736, Euler developed a program for mechanics which had been published and in essential parts elaborated in the two volume treatise on mechanics entitled *Mechanica sive motus scientia analytice exposita* (*Mechanics or the science of motion analytically demonstrated*) [Euler E015/016] written between 1734 and 1736.[2] Projected onto the historical background, Euler initiated a new program in

[1] "The *theory of nature is the science to investigate the causes of the changes which happen to the bodies*. Where a change is, there must be a cause which generates it since it is certain that nothing happens without a sufficient reason. Hence, the essence of the theory of nature is to demonstrate that, supposing that bodies are in certain states, any changes are necessarily caused by the impenetrability which causes just these occurrences of changes." [Euler E842, §§ 1 and 50]

[2] In parallel to mechanics, Euler developed the fundamental concepts in mathematics based on the legacy of Newton and Leibniz and the inventions of Johann und Jakob Bernoulli. The symbol $f(x)$ for a function had been used by Euler for the first time in the paper [Euler E044]. In 1735, Euler wrote an elementary introduction into the *Art of reckoning* [Euler E017] which is distinguished by *consistency* in presenting the basic principles of arithmetics which can be readily acknowledged by

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the development of mechanics whose essence, as it follows from the title, consists in the exclusive application of those analytical methods which had been invented by Newton [Newton, Method of Fluxions] and Leibniz [Leibniz, Nova methodus]. Euler's analytical representation of mechanics is closely correlated with his program for the foundation of the calculus [Euler 1727], [Euler E212] where Euler also favoured an analytical foundation (compare Chap. 5). Although Euler's programs are based on the theories of his predecessors Descartes, Newton and Leibniz, it is essentially different from the Newtonian and the Leibnizian foundation.[3] Not surprisingly, this difference is a common place in the first half of the 18th century[4] since the interpretation of the legacy of Newton[5] was still under debate until 1750[6] whereas later, in the 19th century, a dominant role of Newton had

comparison with the treatise on the same subject published several decades later [Euler E387/388] in 1770. In his preface Euler programmatically stated that the rules for reckoning should be not only demonstrated by application, but also by their foundation by principles. "Dann entweder sind darinn nichts als die blossen Regeln nebst einer grossen Anzahl Exempel enthalten; von dem Grunde aber und den Ursachen, worauf die Regeln beruhen, wird nicht die geringste Meldung gethan." [Euler E017, Preface] Euler provided the reader not only with knowledge, but also with the ability to step forward and to go beyond the commonly accepted state of art. "The reader will not only be enabled to conceive the reason for the rules, but he will be also able to find and invent new rules whose application allows him possible to solve even those problems which cannot be solved by commonly known rules." [Euler E015/016, preface] Euler observed these principles in all his writings which are, therefore, the basis for subsequent extensions and generalizations of the theory as it will be demonstrated in Chaps. 6 and 8.

3 In the end of the 19th century, Euler's inventions in mathematics had been acknowledged by mathematicians and related to the similar development in mathematics called "arithmetization of mathematics". "Vier Stufen werden wir unterscheiden, die durch die Namen Euler, Lagrange, Cauchy und Weierstraß charakterisiert sind. Sie bezeichnen die Hauptmomente in einer Bewegung, die eine immer weiter gehende logische Verschärfung bedeutet und von Herrn F. Klein [Klein, Arithmetization] eine Arithmetisierung der Mathematik genannt worden ist. (...) § 1. Euler schafft den Boden für eine Arithmetisierung. Wenn wir, dem Herkommen folgend, die Periode Eulers als die naive bezeichnen, so sind wir uns der relativen Bedeutung diese Worte wohl bewußt." [Bohlmann] (compare also Klein's review on the development of the concept of function published in 1908 [Klein, Elementarmathematik]). In the 20th century, this essential difference between Euler and his predecessors was also rediscovered as far as Euler's *analytical* approach to mechanics was concerned [Cassirer], [Truesdell], [Simonyi].

4 Altough Newton published the *Principia* already in 1687 (second edition in 1713, third edition in 1726) Newton's physics and mathematics became only popular at the continent with a certain delay. The way for raise of Newtonianism had been paved by Voltaire's *Élémens* [Voltaire, Élémens] published in 1738 and Châtelet's [Châtelet, Institutions] published in 1740. "Of course, not all mathematicians at this time [around 1747] believed Newton's theory, some still believing in Descartes' vortex theories. The announcement that Newton's law was incorrect made many of Descartes' supporters overjoyed and even Euler returned to Descartes' views. Some attacked Clairaut's announcement, for example Buffon who used a metaphysical argument based on the simplicity of the inverse square law." [http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Clairaut.html]

5 In 1710, Johann Bernoulli pointed out that Newton had not proved Kepler's law of ellipses but only its converse and did so himself using calculus, solving "the general problem by reducing it to the same integral that is used to solve it today" [Park, p. 416]. [http://www.sciencetimeline.net/1651.htm]

6 An impression from the main features of the debate can be gained from Maupertuis' *Examen* which is simultaneously a reliable source to get insight in the ranking between the predecessors and contemporaries [Maupertuis, Examen] published in 1756. Maupertuis analyzed the *mechanical*

been established by Mach [Mach, Mechanik]. Following his predecessors Euler's approach is exceptional due to his own method to develop consequent and systematic in parallel both of mathematics and mechanics.[7] Although Euler composed the paper on mathematics *Calculus differentialis* at first in 1727 [Euler, 1727] and at second the *Mechanica* in 1734, it is not only suitable by historical, but also by methodological reasons that we start here with the discourse on the later treatise on mechanics. Thus, the importance of Euler's contribution to physics is emphasized. This order may be also justified by the attempt to analyse the role Euler played as physicist in the 18th century before he become famous as mathematician.

Currently, people are accoustomed to look at the 17th century scholars using the glasses Euler had delvired them by his work on mechanics, but they are often not aware of the true origin of their ability to see the structures of Newtonian mechanics[8] which had only been revealed by the consequent application of the analytical method.[9] Euler revealed the disadvantage of geometrical methods used by Newton and other authors whose treatises are written without the consequent and exclusive application of the analysis, i.e. the calculus or the differential method. As a consequence, the currently accepted representation of *Newton's mechanics* differs considerably from the original version published in 1687. Euler let the reader know somewhat about his own experience in reading Newton's treatise [Euler E015/016,

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principles of Descartes, Newton, Leibniz, Huygens and, finally also of Euler. Even in that time, an appropriate terminology in mechanics was not available and Maupertuis invented the principles of least action which is different from all the principles of his predecessors and contemporaries. The common basis for a unfied theory of mechanics was only step by step developed, in the second half of the 18th century by Lagrange [Lagange, Méchanique analytique] and finally completed in the 19th by Hamilton [Hamilton 1], [Hamilton 2]. Maupertuis analyzed the theories of Descartes (§§ XXVI–XXX), Newton (§§ XXXI–XXXVII), Leibniz (§§ XXXVIII–LII) and the "hypotheses" Huygens (§§ LIII–LVIII). Euler's laws are analyzed in §§ LIX to LXI [Maupertius, Examen]. D'Alembert's contribution is not mentioned. From Maupertuis' summary of the state of art in the mid of 18th century it follows and it is acknowledged within the scientific community, that only Euler introduced a principle which, although it is different from the principles of his predecessors, makes him a coequal follower of the ancestors who is worth to be placed in gallery of "great man" in science ("un aussi grand home") [Maupertuis, Examen].

[7] The parallel development of mathematics and mechanics by Galileo, Descartes, Newton and Leibniz was of great importance for the progress in both disciplines (compare Chaps. 1, 2 and 3).

[8] Only in the end of the 20th century, Chandrasekhar translated the original Newtonian version of the *Principia* into the language based on the calculus. The treatise had been entiled *Newton's Principia for the common reader* [Chandrasekhar]. "Man hat bisher aus den Definitionen VII und VIII und dem Axiom II die Beziehung $K = m\,a$ herausgelesen. Es steht aber damit wie mit den Kleidern des Kaisers im Märchen: jeder sah sie, da er überzeugt war, daß sie da seien, bis ein Kind feststellte, daß der Kaisernichts anhatte. So hat man im einleitenden Kapitel von *Newtons Principia* stets das Axiom ausgesprochen gesehen, daß eine konstante Kraft eine zu ihr proportionale Beschleunigung veursacht (...), wenn man aber die von *Newton* gegebene Grundlegung (...) unter Ausschaltung von allem, was man schon weiß und daher zu finden erwartet, durcharbeitet, so zeigt es sich, daß sie die wichtigste Grundlage für die klassische Mechanik keineswegs enthält" [Dijksterhuis, p. 528, quoted from [Simonyi]]

[9] This development was continued in the 20th century due to the raise of quantum mechanics. Heisenberg based the approach to the new theory on "methods of transcendental algebra" [Schrödinger, Heisenberg]. Nevertheless, also Schrödinger made us of the analytical method and derived a differential equation whose interpretation is related to geometry (compare Chap. 8).

Preface]. Although the reader may be convinced of the truth of the results, he did not obtain sufficient clarity about the theorems and cannot solve problems which differ only slightly in some details from the previously treated model. The reader cannot answer the question by his own efforts if he did not invoke the analysis and did not develop the problems in terms of the analytical method.[10]

> Sed quod omnibus scriptis, quae sine analysi sunt composita, id potissimum Mechanicis obtingit, ut Lector, etiamsi de veritate eorum, quae proferuntur, convincatur, tamen non satis claram et distinctam eorum cognitionem assequatur, ita ut easdem quaestiones, si tantillum immutentur, proprio marte vix resolvere valeat, nisi ipse in analysin inquirat easdemque propositiones analytica methodo evolvat.[11] [Euler E015/016, Preface][12]

The same principles observed Euler in the textbook on *Rechenkunst* or the *Art of Reckoning* [Euler E017][13] written in 1738. According to the general plan, it is not convenient and even not allowed to *modify* the basic principles while studying other

[10] This development introduced by Euler for mechanics (and the foundation of the calculus [Euler E212], compare Chap. 5) had been later successfully continued and completed by Lagrange, Cauchy and Weierstraß. Although Euler did not make use of the concept of *limits*, Weierstraß foundation *by limits* is in goal and spirit not different from Euler's intentions, but had to be regarded as an important step to demonstrate simultaneously the practicability and the advantages of the program since only the arithmetical operations addition, subtraction, multiplaction and division had been accepted as necessary and sufficient for giving demonstrations. In 1895, Klein summarized: "(...) meine Stellung zu derjenigen wichtigen mathematischen Richtung (...) bezeichnen, als deren Hauptrepräsentant *Weierstraß* dasteht, (...) zur *Arithematisierung der Mathematik*. (...) Da zeigte sich bei näherer Untersuchung, daß (...) die Raumanschauung dazu geführt hatte, in übereilter Weise Sätze als allgemeingültig anzusehen, die es nicht sind. Daher die Forderung nach *ausschließlich arithmetischer Beweisführung*." [Klein, Arithmetization]

[11] "That which is valid for all the writings which are composed without the application of analysis is especially true for the treatises on mechanics (*). The reader may be convinced of the truths of the presented theorems, but he did not attain a sufficient clarity and knowledge of them. This becomes obvious if the suppositions made by the authors are only slightly modified. Then, the reader will hardly be able to solve the problems by his own efforts if he did not take recourse to the analysis developing the same theorem using the analytical method." [Euler E015/016, Preface] (*) Newton's *Principia* [Newton, Principia], Hermann's *Phoronomia* [Hermann, Phoronomia]. Euler's procedure is very polite and delicate since he addressed the criticism to the treatise of Hermann before he included other authors. The same advanced art of criticism cultivated Euler in case of the "force of inertia". "(...) atque demonstro leges naturae universales, quas corpus liberum a nulli potentiis sollicitatum observat. (...) habere vim seu facultatem in statu suo perpertuo permanendi, quae nil aliud est nisi ipsa vis inertiae. Minus quidem apte vis nomen huius conservationis causae tribuitur, quia non est homogenea cum aliis viribus proprie sic dictus, cuiusmodi est vis gravitatis, neque cum iis comparari potest; in quo errore plures et imprimis Metaphysici versari solent, vocis ambiguitate decepti." [Euler E015/016, Preface]

[12] Euler continued: "Idem omnino mihi, cum *Neutoni Principia* et *Hermanni Phoronomiam* perlustrare coepissem, usu venit, ut, quamvis plurium problematum solutiones satis percepisse mihi viderer, tamen parum tantum discrepantia problemata resolvere non potuerim." [Euler E015/016, Preface]

[13] The same principles observed Euler also in writing of all his other papers. In the Preface to the book on *Rechenkunst*, the *Art of reckoning*, written in 1735, Euler stated: "Man hat auch im geringsten nicht zu befürchten, dass die Erlernung der Arithmetik auf diese Art schwerer fallen und mehr Zeit erfordern werde, als wann man nur die blossen Regeln ohne einigen Grund vorträgt. Dann ein jeder Mensch begreift und behält dasjenige im Gedächtnis viel leichter, wovon er den Grund und Ursprung deutlich einsieht; und weiss sich auch dasselbe bei allen vorkommenden

system not being mass points. It is only allowed to add some additional principles which are not in contradiction to the previously assumed basic statements on rest and motion. Not surprinsingly, Euler never modified his assumption that the theory of mechanics has to be based on the model of bodies of infinitesimal magnitude, since this assumption is closely connected to his mathematics, especially the foundation of the calculus (compare Chap. 5). The contemporary view on the development of mechanics can be obtained from Gehler's summary [Gehler].[14]

Euler's scientific biography may be naturally related to the decades he spent in St. Petersburg from 1727 to 1741, in Berlin from 1741 to 1763 and after returning to St. Petersburg from 1761 to 1783. In the following table [Euler Archive], mathematical and physically works are listed by whose subjects Euler's almost universal activity in both disciplines is indicated although the reference is far from being complete.[15] The tight correlation between both disciplines in Euler's work is demonstrated from the very beginning and persists in the following decades.

The First Period 1727–1741 St. Petersburg

In St. Petersburg, Euler worked together with the son of Johann Bernoulli. In 1726, Daniel Bernoulli published a paper on the principles of mechanics which represents the state of art in the *analytical* representation of mechanics in the beginning of the 18th century after Newton's *Principia* being the basic treatise for the geometrical representation of mechanics although Newton gave hints that he also applied the calculus [Newton, Principia].[16] In 1684, Leibniz published all

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Fällen weit besser zu Nutz zu machen." [Euler E017, Rechenkunst, Preface] Thirty years later, Euler summarized the development of the methods of the variational calculus and concluded, as he did in analysing the application of geometrical method to mechanics 30 years ago in 1736, that the methods developed by the Bernoullis are only applicable on special problems and had been applied on the isoperimetric problem, the calculation of the brachystochrone and tautochrone. "Johann Bernoulli war vor Erstaunen starr (obstupescas plane) als er der Identität von Brachistochrone, Tautochrone und Zykloide gewahr wurde." [Thiele, Hecht]

[14] In the end of the 18th century, Gehler summarized: "Hermann (Phoronomia s. de viribus & motibus solidorum & fluidorum libri II. Amst. 1716. 4.) trägt die Lehren der höhern Mechanik synthetisch, Euler hingegen (Mechanica, s. motus scientia analytice pertractata. Petrop. 1736. II To. 4. maj. und Theoria motus corporum solidorum s. rigidorum. Rostoch. & Gryphisw. 1765. 4.) analytisch vor. D'Alembert (Traité de Dynamique. à Paris, 1743. 4.) stellt eine sehr scharfe Prüfung der Gründe an, auf welchen das ganze Gebäude der Mechanik beruht, und sucht dieselben mehr aufzuklären und schärfer zu erweisen. Einen ähnlichen Versuch hat auch Lambert gemacht (Gedanken über die Grundlehren des Gleichgewichts und der Bewegung, in den Beyträgen zum Gebrauch der Mathematik, II. Theil, Berlin, 1770. 8. Num. 11.). Kürzere Einleitungen in diese Wissenschaft haben die Herren Kästner (Anfangsgründe der höhern Mechanik. Götting. 1766. 8.) vorzüglich aus Eulers und Joh. Bernoullis Werken, und Karsten (Lehrbegrif der gesammten Mathematik, im 3ten und 4ten Theile) mit schönen Anwendungen auf das Maschinenwesen gegeben. Das neuste System der höhern Mechanik von Herrn de la Grange (Mechanique analytique. à Paris, 1788. 8.) leitet in der höchsten Allgemeinheit, und ohne alle Figuren, die ganze Statik und Dynamik aus einer einzigen Grundformel ab." [Gehler, Mechanik]

[15] For the complete listing of Euler's work see The Euler Archive [http://math.dartmouth.edu/∼euler/].

[16] Especially discussing the "first and ultimate ratios" [Newton, Principia].

fundamental rules the calculus is based on without an application to mechanics [Leibniz, Nova methodus]. Euler decided to represent mechanical principles using the Leibnizian version and developed his program for mechanics [Euler E016/17], [Euler E022]. Although Euler became later famous as a mathematician his first comprehensive treatise was on physics. Euler introduced mathematical principles, developed and applied algorithms [Euler 1727] whose foundation were later demonstrated in detail [Euler E101], [Euler E102], [Euler E212], [Euler E342], [Euler E366], [Euler E385]. Furthermore, Euler wrote an elementary introduction in the art of reckoning for pupils [Euler E017]. The full list of Euler published writings comprises until now 862 items. The following papers are selected from the collection presented in the Euler Archive [Euler Archive]. The review demonstrates in short the scientific activities of Euler since only the main writings being of interest are listed according to the data written.

1727
E02 Dissertatio physica de sono, presented 1727, published 1727.

1731
E022 De communicatione motus in collisione corporum, presented 1730, published 1738.

1734
E015 Mechanica sive motus scientia analytice exposita, volume 1 presented 1736, published 1736.

E044 De infinitis curvis eiusdem generis seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis, presented 1734, published 1740.

1735
E017 Rechenkunst, presented 1738, published 1738.

1736
E016 Mechanica sive motus scientia analytice exposita, volume 2 presented 1736, published in 1736.

The Second Period 1741–1766 Berlin

The next extraordinary book Euler wrote was the *Methodus inveniendi lienas curvas* [Euler E065] which may be also considered as resulting from correlated development of mathematical and mechanical principles. In the Addimenatum II, Euler discussed the derivation of the equation of motion from a general principle which became later known as the principle of least action.
Between 1760 and 1762, at the end of his Berlin period Euler wrote the *Letters to a German princess* which had been later published as a three volume book [Euler E343], [Euler E344], [Euler E417].[17]

[17] There are 234 letters. The first letter was written on April 19, 1760 and it is on *Extension*, the last one on May 1762.

1743
E065 Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti,[18] presented 1744, published 1744.

1744
E165 De motu corporum flexibilium, presented 1744, published 1751.
E174 De motu corporum flexibilium, presented 1751, published 1751.

1745
E077 Neue Grundsätze der Artillerie, presented 1745, published 1745.
E101 Introductio in analysin infinitorum, volume 1 presented 1748, published 1748.
E102 Introductio in analysin infinitorum, volume 2 presented 1748, published 1748.

In *Introductio in analysin infinitorum* [Euler E101], Euler introduced the concept of function [Thiele 2007]. In the treatise *Anleitung zur Naturlehre* [Euler E842] Euler developed systematically the basic concepts of mechanics which had been already assumed as the conceptual basis of the science of motion analytically demonstrated [Euler E015/016].[19] Euler rejected explicitly the concept of a "force of intertia" and based the theory of rest and motion on the *relative motion of bodies* whose behaviour is analyzed by *observers* (Zuschauer) in dependence on their different perspectives. Euler analyzed and rejected the Leibniz-Wolffian theory of monads by demonstrating exclusively its *logical* inconsistency. Therefore, accepting Euler's approach neither a mechanical nor mathematical nor metaphysical argumentation is needed.[20] The theory will be rejected because of its contradictory assumptions.

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[18] "The methods which the Bernoulli brothers developed to solve the challenge problems they were tossing at each other were put in a general setting by Euler in *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti* published in 1744. In this work, the English version of the title being *Method for finding plane curves that show some property of maxima and minima*, Euler generalises the problems studies by the Bernoulli brothers but retains the geometrical approach developed by Johann Bernoulli to solve them. He found what has now come to be known as the Euler-Lagrange differential equation for a function of the maximising or minimising function and its derivative." [http://www-groups.dcs.st-and.ac.uk/∼history/HistTopics/Brachistochrone.html]

[19] In the three books, *Mechanica* [Euler E015/016] (1736), *Anleitung* (1744–1748) [Euler E842] (probabably written between 1744 and 1748, published only 1862) *Theoria* [Euler E289] (1765), Euler discussed the basic principles of mechanics. The papers [Euler E149] on *space and time*, [Euler E177] on a *new principle of mechanics*, [Euler E181] on the *origin of forces* and [Euler E197] on the *principles of least action* are to be related to these comprehensive representations and application of mechanical principles.

[20] Euler's contemporaries reported that he was not inclined to take part in discussions on metaphysical questions. Euler himself declared: "Cuius rationii vis, etiamsi nondum satis perspiciatur, tamen, quia cum veritate congruit, non dubito, quin ope principiorum sanioris Metaphysicae ad maiorem evidentiam evehi queat; quod negotium aliis, qui Metaphysicam profitentur, relinquo (Obgleich nun die Kraft dieses Schlusses noch nicht genügend einleuchtet, zweifele ich nicht daran, daß es möglich sein wird, ihn mit Hilfe einer gesunden Metaphysik besser zu begründen. Dies ist aber eine Sorge, die ich anderen, die sich mit Metaphysik befassen, überlasse)." [Euler E065, Additamentum II].

1744–1748
[E842] Anleitung zur Naturlehre, Instruction for natural science published posthumously.

1746
E081 Gedancken von den Elementen der Cörper, presented 1746, published 1746.
E173 Nova methodus inveniendi trajectorias reciprocas algebraicas, presented 1751, published 1751.

1747
E112 Recherches sur le mouvement des corps célestes en général, presented 1747, published 1749.
E854 Différentes pièces sur les monades, presented 1862, published 1862.

1748
E149 Réflexions sur l'espace et le tems, presented 1748, published 1750.
E212 Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum, volume 1 presented 1755, published 1755.

1749
E146 Réflexions sur quelques loix générales de la nature qui s'observent dans les effets des forces quelconques, presented 1748, published 1750.

1750
E177 Découverte d'un nouveau principe de Mécanique, presented 1750, published 1752.
E181 Recherches sur l'origine des forces, presented 1750, published 1752.
E230 Elementa doctrinae solidorum, presented 1752, published 1758.
E232 De motu corporum coelestium a viribus quibuscunque perturbato, presented 1752, published 1758.

1751
E260 Tentamen theoriae de frictione fluidorum, presented 1756, published 1761.
E292 Du mouvement de rotation des corps solides autour d'un axe variable, presented 1758, published 1765.

1752
E197 Harmonie entre les principes généraux de repos et de mouvement de M. de Maupertuis, presented 1751, published 1753.
E199 Examen de la dissertation de M. le Professeur Koenig, inseree dans les actes de Leipzig, pour le mois de mars 1751, presented 1751, published 1753.

1753
E198 Sur le principe de la moindre action, presented 1751, published 1753.
E200 Essay d'une démonstration métaphysique du principe général de l'équilibre, presented 1751, published 1753.
E225 Principes généraux de l'état d'équilibre des fluides, presented 1755, published 1757.
E258 Principia motus fluidorum, presented 1756, published 1761.

1755

E226 Principes généraux du mouvement des fluides, presented 1755, published 1757.

E227 Continuation des recherches sur la théorie du mouvement des fluides, presented 1755, published 1757.

1756

E296 Elementa calculi variationum, presented 1764, published 1766.

E297 Analytica explicatio methodi maximorum et minimorum, presented 1764, published 1766.

1758

E336 Du mouvement d'un corps solide quelconque lorsqu'iltourne autour d'un axe mobile, presented 1760, published 1767.

1759

E285 Investigatio functionum ex data differentialium conditione, presented 1762, published

E305 De la propagation du son, presented 1759, published1766.[21]

E306 Supplément aux recherches sur la propagation du son, presented 1759, published 1766.

E307 Continuation des recherches sur la propagation du son, presented 1759, published 1766.

E308 Recherches sur le mouvement de rotation des corps, célestes presented 1759, published 1766.

1760

E289 Theoria motus corporum solidorum seu rigidorum, presented 1765, published 1765.

E343 Lettres à un princesse d'Allemagne sur divers sujets de physique & de philosophie, presented 1768, published 1768.

1761

E344 Lettres à un princesse d'Allemagne sur divers sujets de physique & de philosophie, presented 1768, published 1768.[22]

[21] Alternatively to Newton's theory of light who assumed that particles are emitted by the sun, Euler developed a wave theory. The light is progating in the ether like sound waves are propagating in the air [Euler E343]. "III. Of Sound, and its Velocity, XVII. Of Light, and the systems of Descartes and Newton, XVIII. Difficulties attending the System of Emanation, XIX. A different System respecting the Nature of Rays and of Light proposed, XX. Of the propagation of Light, XXI. Digression on the Distances of the Heavenly Bodies, and on the Nature of the Sun, and his Rays." [Euler E343, New York 1823, Third Ed. 1837]

[22] "The Letters of Euler to a German Princess have acquired, over all Europe, a celebrity, to which the reputation of the Author, the choice and importance of several subjects, and the clearness of elucidation, justly entitle them. They have deservedly been considered as a treasure of science, adapted to the purposes of every common seminar of learning. (...) they convey accurate ideas respecting a variety of objects, highly interesting in themselves, or calculated to excite a laudable curiosity; they inspire a proper taste for the sciences, and for that sound philosophy which, supported by science, and never loosing sight of her cautions, steady, methodical advances, runs no risk of perplexing or misleading the attentive student." [Condorcet, Letters]

The Third Period 1766–1783

1762
E417 Lettres à une princesse d'Allemagne sur divers sujets de physique & de philosophie Tome troisième A Saint Petersbourg de l'imprimerie de l'academie imperiale des sciences MDCCLXXII, presented 1771.

1763
E385 Institutionum calculi integralis volumen tertium, presented 1770, published 1770.

1767
E387 Vollständige Anleitung zur Algebra, presented 1770, published 1770.
E388 Vollständige Anleitung zur Algebra, book 2 presented 1770, published 1770.

The papers of Euler had been enumerated and listed by Eneström who introduced the classification which currently is mainly used [Euler Archive]. Beside the Eneström index there is an earlier version introduced by Fuess [Euler Archive].

300 Years Later. Tercentenary Celebrations in 2007
The Tercentenary had been celebrated all over the world, especially in those places Euler was living and working. Basel [Tercent Basel], Berlin [Tercent Berlin] and St. Petersburg [Tercent StPeter].[23]

Bradley R E and Sandifer C E (eds) (2007) Leonhard Euler 5. Life, Work and Legacy. Elsevier, Amsterdam [Bradley, Sandifer]
Dunham W (ed) (2007) Genius of Euler: Reflections on his Life and Work. Mathematical Association of America [Dunham, Euler][24]
Sandifer C E (2007) How Euler Did It. Mathematical Association of America [Sandifer, How][25]
Euler and Modern Science. Boglyubov N N, G. K. Mikhauilov G K, and A. P. Yushkevich A P (eds) (translated from the Russian) Mathematical Association of America
Euler at 300: An Appreciation. Bradley R E, D'Antonio L A and C. Edward Sandifer C E (eds) Mathematical Association of America [Bradley, D'Antonio, Sandifer]

4.1 Euler's Program for Mechanics

The first papers Euler published are on the isochrone and brachistochrone[26] problems [Euler E001], [Euler E003] and the propagation of sound [Euler, E002].[27]

[23] See also [Tercent MAA].

[24] Dunham W (1999) Euler: The Master of Us All. Washington (DC) [Dunham]

[25] See also [Sandifer], [Sandifer, Eearly].

[26] The brachistochrone problem was posed byJohann Bernoulli in *Acta Eruditorum* in June 1696. [http://www-groups.dcs.st-and.ac.uk/∼history/HistTopics/Brachistochrone.html]

[27] "Later, Euler continued the investigation of the propagation of sound [Euler E305], [Euler E306], [Euler E307] (see also [Euler E344]). Based on the studying of sound and following

The first book Euler published is on mechanics entitled *Mechanica sive motus scientia analytice exposita* [Euler E015/016].[28] The analytical demonstration is nothing else but the application of the calculus in the Leibnizian notation to the formulation and solution of geometrical and mechanical problems.[29] Euler made use of the method in the papers [Euler E005], [Euler E008], [Euler E009] and [Euler E013] written between 1727 and 1734. As it can be seen from Eneström index, the following treatise was already on *Mechanics* [Euler E015/016]. In the *Mechanica*, the basic principles of mechanics and mathematics had been used, but only later thouroughly elaborated and founded more in detail in the papers

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Huygens, Euler developed a wave theory of light which was in opposite to Newton's theory [Euler E344]. In 1768, Euler proposed that the wave length of light determines its color. (i) Euler's anti-Newtonian theory of light as wave propagation. (ii) For example, although Isaac Newton had declared it theoretically impossible to produce achromatic lenses, Euler disagreed based on the fact that the eye is composed of lenses that can create a near-perfect image." [http://www.ingentaconnect.com/content/tandf/tasc/1988/00000045/00000005/art00005]

[28] "Euler's early papers mathematical papers show the influence of Johann Bernoulli, his mentor. E001 and E003 are concerned with the isochrone and brachistochrone problems. E002 is a dissertation on sound (...); E004 Euler uses mechanics expressed mathematically to design ships. He adopts Newton's Law of Resistance (which says that the pressure exerted by a fluid acting against a plane surface is proportional to the square of the speed, neglecting back pressure) by interpreting it as a statement about differential elements of surface. E005 is concerned with curves that intersect orthogonally, a much longer paper, [Euler applied the calculus in Leibnizian notation and the concept of function] (...). E006 is concerned with the involute of the circle, formed by unwinding a string along the tangent, as a correcting mechanism for the period of a chronometer. E007 is an attempt to explain atmospheric phenomena in terms of air vesicles, fine matter, and centrifugal force. E008 is about the general solution of heavy planar curves under various loadings, catenaries, sails, etc. (Another blend of analysis, statics, geometry, and algebra is used here in this interesting paper, in which the Calculus has definitely 'Come of Age', as it were. Heavy curves associated with chains, ropes, sails, strings, etc, are analysed both with and without elasticity, after a general formula has been found from statics principles). E009 is a classic early paper on the shortest curve joining two points on a surface [Application of the calculus in Leibnizian notation]. E010 is the start of Euler's love affair with the exponential function, related to easing the pain of solving differential equations [Application of the calculus in Leibnizian notation]. E011 is a later paper, and relies on previous work not yet covered in this series of translations. E012 is a masterful work, in which Euler first establishes a surprisingly simple geometric condition for tautochronic curves, and then shows how to generate such curves, both analytic and algebraic, starting from the familiar cycloid; E013 extends the analysis to a resistive medium where the resistance is in proportion to the square of the speed [Application of the calculus in Leibnizian notation to a mechanical problem]. E014 is an elementary treatment of finding the pole star from three measurements on a star over time." [Euler Archive, Euler: Some early paper] [comments in brackets, D.S.]

[29] The basic principles of the calculus had been published by Leibniz in 1684 [Leibniz, Nova Methodus] and by Newton in 1704 and 1710 [Newton, Quadrature (Harris)] (compare Chap. 3). Soon after Leibniz's treatise appeared, Johann and Jakob Bernoulli made use of the new method. "One of the most significant events concerning the mathematical studies of Jacob Bernoulli occurred when his younger brother, Johann Bernoulli, began to work on mathematical topics. (...) Jacob Bernoulli was appointed professor of mathematics in Basel in 1687 and the two brothers began to study the calculus as presented by *Leibniz* in his 1684 paper on the differential calculus (...). They also studied the publications of *von Tschirnhaus*. It must be understood that Leibniz's publications on the calculus were very obscure to mathematicians of that time and the Bernoullis were the first to try to understand and apply Leibniz's theories." [Bernoulli, http://www-history.mcs.st-andrews.ac.uk/Biographies/Bernoulli_Jacob.html]

[Euler E065], [Euler E149], [Euler E177], [Euler E181], [Euler E197], [Euler E320] and [Euler E017], [Euler E101], [Euler E102], [Euler E149], [Euler E212], respectively. Hence, the reading of Euler's treatise was a challenge for the contemporary reader in the 18th century[30] since he had to be familiar with the main developments in mathematics and physics initiated by Newton and Leibniz whose theories had been discussed in the first half of the 18th century in parallel. Euler merged the Newtonian and Leibnizian principles of mechanics and the *Leibnizian representation of the calculus* and could make use of the applications and developments elaborated after 1684 mainly by Johann[31] and Jakob Bernoulli, L'Hospital, Varignon and Daniel Bernoulli.[32] To increase the intricacy of understanding, the reader had to be also familiar with the interdependence of mathematics and mechanics with the metaphysical parts of the systems of Descartes, Newton[33] and Leibniz.[34] Euler's treatise on mechanics is also the result of a critical reading of

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[30] The situation was just opposite to present times where nobody is accustomed to make use of Newton's geometric method.

[31] Johann Bernoulli's treatises on integral and differential calculus appeared in 1742.

[32] The difficulty of the problem can be estimated by the fact that neither the Bernoullis nor Varignon or other authors presented such textbook before Euler it did. The novelty of Euler's approach may be acknowledged and exemplified by the similar development taking place in the 20th century as the young Heisenberg invented the novel principles of quantum mechanics. The former generation of physicist represented by Planck, Einstein, Bohr, Sommerfeld, von Laue and Born, as Newton and his followers 200 years ago, was accustomed to be involved in the *geometrical interpretation of mechanics* based on the motion of bodies along trajectories in space whereas Euler and Heisenberg *relied on states* (of the classical or quantum mechanical system) based on the concept of mass (and energy [Heisenberg 1925]) as a *numerical* parameter (compare Jammer [Jammer, Mass] and Couturat on Leibniz's conservation of living forces [Couturat, Cassirer]). As the concept of mass, the concept of (energetic) states is related only to numbers or, generally to integers if the stats are assumed to be *countable*. As a consequence, all quantities are divided into the two classes of (i) *countable* and (ii) *uncountable* quantities. Following Euler, the notion of path is based on the model of an uncountable number of intermediate places. "Es wäre ungereimt, die Mittelorte zählen zu wollen." [Euler E842, § 19] The path consists of a number of unassignable places which are reduced to an assignable number by measurement. Hence, the geometrical representation of motion is based on the arithmetical distinction between a *countable* and an *uncountable* number of places the body traversed in motion. The distance between neighboured points is indeterminate, but it has to be ensured that the continuity of motion is not disturbed or destroyed.

[33] As it was mentioned above (compare Chaps. 2 and 3) and can be read from the title, Newton intended to develop the *Mathematical Principles of Natural Philosophy* and the contemporaries understand and interpreted his mathematical methods and physical theories to be correlated to the debate on metaphysical and religious questions (compare Berkeley's treatise entitled *The Analyst; or, a Discourse Addressed to an Infidel Mathematician* (1734), [Berkeley; Analyst]). Furthermore, the debate was running on the background formed by the competition between France, England and Germany [Châtelet, Institutions]. Nowadays, Newton's mathematics and physics had been reconsidered by embedding in the context of the *complete legacy* and the interdependence of its different parts had been studied [Keynes], [Westfall, Never], [Snobelen].

[34] "In 1713 Johann became involved in the Newton-Leibniz controversy. He strongly supported *Leibniz* and added weight to the argument by showing the power of his calculus in solving certain problems which *Newton* had failed to solve with his methods. Although Bernoulli was essentially correct in his support of the superior calculus methods of Leibniz,

Descartes, Newton and Leibniz promoted by the scientific supervision by Johann and Daniel Bernoulli.[35]

4.1.1 Geometry and Motion

Following Newton and Leibniz, mechanics results from the completion of geometry by time and forces (compare Chap. 2). Following Euler, mechanical laws describe the relations between *bodies* and *space* completed by the relations *between different bodies* occupying distinguished places. Hence, a path or the trajectory described by a moving body is always related to the 3D space or the full space. Different kinds of motion result from the relations to (i) the full space (3D) or (ii) the subspaces of two (2D) or one (1D) dimensions [Euler E289, § 12]. The supposition of the space or the extension is an essential part of Descartes' legacy. Hence, the position of a body is always related to all dimensions of the space, i.e. the position is geometrically complet determinate.[36] Any motion is assumed to be performed in the 3D space and it is, for the purpose of analytical descripion, subsequently projected onto subspaces as long as no further projection is possible. Time did not correlate the different directions the space is decomposed since it is not only assigned to motion from the very beginning, but the time ordering is also preserved for the projection. In Newton's representation, the fluents and fluxions are time dependent quantities (compare Chaps. 2 and 3) and lines, surfaces and solids are subsequently generated by motion of points, lines and surfaces, respectively [Newton, Quadrature]. Euler's mechanics is also contrasting Leibniz's mechanics since Leibniz considered the velocity and the motion rather as a property of single bodies caused by forces than as purely geometric relations between bodies. Leibniz introduced a relational definition of forces assuming *derivative* forces between *interacting* bodies which correspond to Newton's *impressed moving* forces if each of the bodies is considered independently of the other one [Newton, Principia]. However, Leibniz introduced also *primitive* forces being of non-relational origin. Therefore, in Newton's and Leibniz's mechanics coexist different schemes since relational and non-relational

he also supported Descartes' vortex physics on the Continent." [http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Bernoulli_Johann.html]

[35] Daniel Bernoulli published the *Hydrodynamica* in 1738 (written between 1735 and 1738) [Daniel Bernoulli, Hydrodynamica]. His interest in hydrodynamics was stimulated by medical question concerning the flow and the pressure of blood. "Two years later he pointed out for the first time the frequent desirability of resolving a compound motion into motions of translation and motions of rotation. His chief work is his *Hydrodynamique* (*Hydrodynamica*), published in 1738; it resembles Lagrange's *Méchanique Analytique* in being arranged so that all the results are consequences of a single principle, namely, in this case, the conservation of energy." [Daniel Bernoulli, Wikipedia] Euler published a general principle in 1752 (written in 1750) [Euler E177].

[36] Geometrically with respect to the space, the straight line is correlated with a plane oritented perpendicular to the line and, vice versa, a plane is correlated with a straight line oriented perpendicular to the plane. Mechanically, the homogeneity of uniform motion along the straight line results in a homogeneity of all planes oriented parallel to any chosen plane oriented perpendicular to the straight line [Euler E842].

versions for the definition of forces are used simultaneously. This ambiguity was finally removed by Euler.

Euler introduced a rigorous and complete *relational theory of motion and forces*. The relational theory of motion is formulated for force-free or uniform motion of bodies along a straight direction. Here, *geometry* is *necessary* and *sufficient* to describe the relations between bodies resting or moving *relatively* to each other (compare Chap. 6). This part of mechanics cannot be derived from static or the *science of equilibrium* since the equilibrium [Leibniz, Specimen, I (8)] is always established between forces of equal or different magnitude. Euler established this essential part of mechanics which is independent of the theory of forces. The bodies are only governed by *internal* principles whereas the forces are related to *external* principles [Euler E389] (compare Chap. 6). Motion and rest are defined operationally, i.e. by the preservation or the change of the *geometrically* defined *relative distances* between bodies[37] [Euler E015/016, §§ 7 and 97], [Euler E842]. Leibniz defined rest non-operationally as an "evanescent motion" [Leibniz, Specimen, II (4)] where the geometrical relations between bodies remain to be indeterminate.

Following Euler, the mass of a body is also defined operationally by the interaction of bodies [Euler E181, §§ 39–42], [Euler E289, §§ 137–144] (for the operational definition of the mass by Euler compare [Jammer, Mass]) in the same manner as it was later proposed by Mach[38] for the relation between masses and accelerations. The relation between the accelerations and the masses follows from the conservation of momentum, in Euler's derivation from the equality of the magnitude of forces generated in the interaction of bodies which are oriented in opposite directions [Euler E181, §§ 39–42], [Euler E289, §§ 150–155].

Euler rejected all forces except those which have been called *derivative* forces by Leibniz [Leibniz, Specimen, I (4)]. These forces are generated during the interaction by the interacting bodies whre the bodies form a system ("coeunt") [Euler E289, § 131]. The forces are equal in magnitude and opposite in direction (compare Newton's 3rd Law [Newton, Principia]).

Euler's relational theory of mechanics is crowned by the introduction of observers who are associated with bodies and who are either resting or moving relatively to each other as far as the bodies perform these motions. The action of the observers is the measurement of distances between bodies and the comparison of results of their measurements [Euler E842, §§ 77–83] (compare Chap. 6).

Euler based his mechanics on Newton's *Principia*. Reading Euler's book one can be confused and astonished since Euler considered, in contrast to Newton, *absolute* and *relative* motion *theoretically* on an equal footing. Moreover, Euler considered space and time on equal footing whereas Newton based mechanics and the

[37] Newton's assumption of absolute places of bodies in absolute space may be considered as non-operational determination since any measurement is excluded (compare the criticism by Mach [Mach, Mechanik]).

[38] "Die wahre Definition der Masse kann nur aus den dynamischen Beziehungen der Körper zueinander abgeleitet werden", i.e. from the relation between the accelerations of at least two bodies. [Mach, Mechanik, pp. 239–242].

calculus on the priority of time. The mechanical quantities are represented by fluents and fluxions where, using a contemporary terminology, both types of quantities are functions of time. Newton considered the mechanical and mathematical quantities as being generated by a continual flux (compare Chap. 3). Following Euler, time and space may be chosen and, moreover, have to be chosen as independent variables to demonstrate the appearance of two measures related to motion of a body of mass m, first the *Cartesian* measure mv and second the *Leibnizian* measure $mvv/2$, nowadays called momentum and kinetic energy, respectively. However, having introduced absolute time and absolute space, Newton also stated:

> And so, instead of absolute places and motions, we are using relative ones; and that without any inconvenience in common affairs. [Newton, Principia][39]

Euler did more, he demonstrated that both the descriptions are *equivalent* so that we are not only *able* but we are also *allowed* to use both representations. The basic statement is that the mechanical laws for absolute and relative motions are equivalent as far as the theory of the motion of bodies is concerned. However, Euler also demonstrated the *difference* which appears after introducing the relation to experience. Finally, the introduction of observers destroys the theory of absolute motion [Euler E842, §§ 77–83]. Relative motion can always be related to observation and can be determined experimentally, absolute motion cannot [Euler E015/016, §§ 7 and 80]. It is impossible for us to develop a determinate concept (imagination) of the infinite immeasurable space[40] and its boundaries. Therefore, *instead* of both these concepts, and not merely "without any inconvenience in common affairs", as Newton stated,[41] we consider a respective space, i.e. a limited finite space and

[39] "Noch ein weiteres Charakteristikum zeichnet die dritte Version von *De motu* aus: das auf das Problem 5 folgende Scholium über 'den ungeheuren und vollkommen unbeweglichen Himmelsraum'. Wie die Idee der inhärenten Kraft von Körpern, die er für ein spezifisches Merkmal echter Bewegung hielt, hing diese Scholium mit Newton's Abscheu vor dem Relativismus von Descartes' Physik zusammen. In seiner Schrift *De gravitatione* hatte er früher gesagt, die größte Absurdität des Kartesianischen Relativismus liege in der Konsequenz, 'daß ein sich bewegender Körper keine bestimmte Geschwindigkeit und keine bestimmte Bahn, auf der er sich bewegt, besitzt'. (...) In Bezug auf die Bewegung mochte er sich Descartes unterworfen haben, doch was den Relativismus anging, der für Newton stark nach Atheismus roch, leistete er bis zu seinem letzten Atemzug Widerstand." [Westfall, Newton, pp. 213–214]

[40] Note the difference Euler made between an *infinite* and an *immense* or *immeasurable* space. The immense or immeasurable space cannot be related to experience [Euler E015/016, § 80]. It is also impossible to relate this space to mathematics. In contrast, infinity is related to mathematics. Therefore, assuming that mechanics has to be related to experience and mathematics, it is finally necessary to exclude the absolute space from mechanics. Euler started this procedure with the exclusion and rejection of *absolute* motion [Euler E842], [Euler E289], but preserved the absolute space. However, the absolute quantities are not related to geometry, but to the inertia of bodies. Therefore, one may interpret this behaviour as a motion along a geodetic line in Euclidean space.

[41] Newton added: "(...) but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. It may be that there is no body really at rest, to which the places and motions of others may be referred." [Newton, Principia, Definitions]

boundaries related to bodies (corporeal boundaries), and consider motion and rest with respect to this space and its boundaries [Euler E015/016, § 7].

Assuming only relative or respective rest and motion, we conclude from *relative* motion that the *same* laws for rest and motion are also true for absolute motion. Euler explained this assumption, however, in a sophisticated manner, since he claimed that the laws of relative motion are not sufficient to determine the direction and the magnitude of absolute motion[42] [Euler E015/016 § 80]. However, considering the global plan of his treatise which is mirrored in the table of contents referred below, it follows quite naturally that relative motion is of greater importance as absolute motion since *all* parts of mechanics are based on relative motion.[43]

Euler considered (i) the general laws of motion, (ii) the effect of forces, (iii) the motion along a straight line, (iv) the motion along a straight line in a resisting medium, (v) the motion on curved lines and (vi) the motion on curved lines in a resisting medium. (i) De motu in genere, (ii) De effectu potentiarum, (iii) De motu rectilineo, (iv) De motu rectilineo puncti liberi in medio resistente, (v) De motu curvilineo and (vi) De motu curvilineo puncti liberi in medio resistente [Euler E015/016]. Obviously, the motion of a point in a *resisting* medium cannot be described in another way except as *relative* motion. Therefore, the basic principles of relative motion have to be developed in advance, i.e. for the free motion of a point representing a body in a non-resisting medium. The model for a non-resisting medium is the vacuum.

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[42] Euler claimed that magnitude and direction cannot be determined experimentally in an absolute space. Therefore, both the concepts had to be excluded from theory. Obviously, Euler used the same argumentation which had been later renewed by Mach and Heisenberg. The difference to Mach is that Mach followed the *Newtonian* ordering scheme of time, space and motion and tried to remove the concept of absolute space and absolute motion [Mach, Mechanik, pp. 233], whereas Euler started with the rejection of absolute motion. The essential difference is that time can be considered either as dependent on or independent of motion (compare Sect. 4.3). Like Euler, Mach argued that absolute space cannot be measured [Mach, Mechanik, pp. 216]. Mach published the treatise on Mechanics in 1883 and mentioned the stimulating reading of the treatises of Lagrange (*Analytical mechanics*), Jolly (*Prinzipien der Mechanik*, published 1852), Dühring (*Kritische Geschichte*, published 1873), Kirchhoff (*Vorlesungen über mathematische Physik*, 1874) and Helmholtz (*Die Tatsachen in der Wahrnehmung*, published 1879) [Mach, Mechanik, Preface]. However, Mach did not mention Euler's *Mechanica* and *Theoria* altough both books were translated into German and published by Wolfers in 1848 (*Mechanica*) and 1853 (*Theoria*). Moreover, the treatise *Anleitung* had been published in 1862, i.e. 20 years before Mach published his book. Mach stressed the difference of his treatise to the analysis of Lange. "Die Differenz, die noch besteht und vielleicht bestehen bleiben wird, liegt darin, daß Lange als *Mathematiker* an die Frage herangetreten ist, während ich die *physikalische* Seite ins Auge gefaßt habe." [Mach, Mechanik, p. 235] The difference and oppositions between these points of view did not appear in Euler's analysis since Euler considered *mathematics and mechanics* simultaneously.

[43] The theory of torque is also considered as a special case of *relative* motion, e.g. the motion of a body relative to a fixed point or a fixed axis [Euler E289].

4.1.2 Euler's Program for Mechanics: Mechanica and the Arithmetization of Mechanics

Euler replaced the geometrical representation of mechanics by the analytical representation using the calculus in the Leibnizian representation. The basic facts observed for all kinds of bodies are the preservation of the states of rest and motion due to the inertia of bodies. The principles are subdivided into *internal* and *external* principles. This classification had been introduced in the *Mechanica* [Euler E015/016], maintained in the *Anleitung* [Euler E842] and explicitly elaborated in the *Theoria* [Euler E289].[44]

According to the general plan, it is not convenient and even not allowed to *modify* the basic principles while studying other systems not being made up of mass points. It is only allowed to add some additional principles which are not in contradiction to the previously assumed basic statements on rest and motion. Not surprisingly, Euler never modified his assumption that the theory of mechanics has to be based on the model of bodies of infinitesimal magnitude, since this assumption is closely connected to his mathematics, especially the foundation of the calculus.[45]

> Those laws of motion which a body observes when left to itself in continuing either rest or motion pertain properly to infinitely small bodies, which can be considered as points.[46] (...) The diversity of bodies therefore will supply the primary division of our work. First indeed we shall consider infinitely small bodies (...). Then we shall attack bodies of finite magnitude which are rigid. (...) Thirdly, we shall treat of flexible bodies. Fourthly, of those which admit extension and contraction. Fifthly, we shall subject to examination the motions

[44] In 1720 and later in the 1720's and 1730's Euler started and continued his scientific career under the supersion of Johann Bernoulli in Switzerland and in St. Petersburg, respectively, the scientific background at his disposal was not only formed by the rapidly developing mathematics, but also by the controversies in physics and philosophy between the Cartesian, Newtonian and Leibnizian schools. In 1738, Voltaire summarized the that state of affairs: "Descartes, sans faire mention de la force, avançait sans preuve qu'il y a toujours quantité égale de mouvement; et son opinion était d'autant moins fondée que les lois mêmes du mouvement lui étaient absolument inconnues. Leibnitz, venu dans un temps plus éclairé, a été obligé d'avouer, avec Newton, qu'il se perd du mouvement; mais il prétend que, quoique la même quantité de mouvement ne subsiste pas, la force subsiste toujours la même. Newton, au contraire, était persuadé qu'il implique contradiction que le mouvement ne soit pas proportionnel à la force." [Voltaire, Élémens, Chap. IX] Space, time and matter are discussed in the First Part *On Metaphysics*: "Première Partie. Métaphysique. Chapitre II. De l'espace et de la durée comme propriétés de Dieu. Sentiment de Leibnitz. Sentiment et raisons de Newton. Matière infinie impossible. Épicure devait admettre un Dieu créateur et gouverneur. Propriétés de l'espace pur et de la durée." [Voltaire, Élémens] Voltaire acknowledged the communication with Madame du Châtelet who published her own analysis and summary in 1740 [Châtelet, Institutions].

[45] "Während Newton in Bewegungsabläufen denkt und Leibniz seine Differentiale anhand von Kurven und Tangenten erklärt, löst sich Euler vollständig von mechanischen und geometrischen Vorstellungen." [Euler E101/102, (Preface of Wolfgang Walter, Einführung zur Reprintausgabe)]

[46] "98. Istae motus leges, quas corpus sibi relictum vel quietem vel motum continuando observat, spectant proprie ad corpora infinite parva, quae ut puncta possunt considerari." [Euler E015/016, § 98]

> of several separated bodies, some of which hinder [each other] from executing their motions as they attempt them. Sixthly at last, the motion of fluids will have to be treated. [Euler E015/016, §98][47]

Euler consequently constructed mechanics on the basic concept of (i) rest and motion,[48] (ii) the model of an infinitely small body and (iii) on the basic distinction between internal and external principles. The internal principles are related to the general property of all bodies called inertia and impenetrability. Following Newton [Newton, Principia], the basic distinction is made between absolute and relative motion and almost all theorems are checked whether they are to be valid either for absolute or relative motion or for absolute and relative motion. However, already in the first chapter *On motion in general*, Euler made a decision on favour of *relative* motion[49] [Euler E015/016, §§ 7 and 97] (compare Chap. 6). The basic distinction is made between (i) rest, (ii) uniform motion and (iii) and non-uniform motion. In case of a non-uniform motion the "smallest elements of the path are able to be conceived as equably traversed" by the body. Euler's program for mechanics turned out to be the reliable basis not only for the development in mechanics in the 18th century, but also in the 19th and 20th centuries.[50]

In 1788, Lagrange summarized the results obtained by the mathematicians and physicist in the 18th century in the treatise programmatically entitled *Mécanique analytique* [Lagrange, Mécanique] establishing a new representation of mechanics which may be called *arithemetization of mechanics*.[51] Lagrange completed the 100 years lasting development of mechanics which took place between the mid of the 17th and the mid of the 18th centuries. In 1687, Newton was successful to demonstrate the unification of mechanics, i.e. the unification of statics and dynamics, based on geometrical methods. The principles of statics had already been introduced by the ancient scholars. The replacement of geometrical representation used by Newton by arithmetical methods had been consequently continued and completed by Euler and

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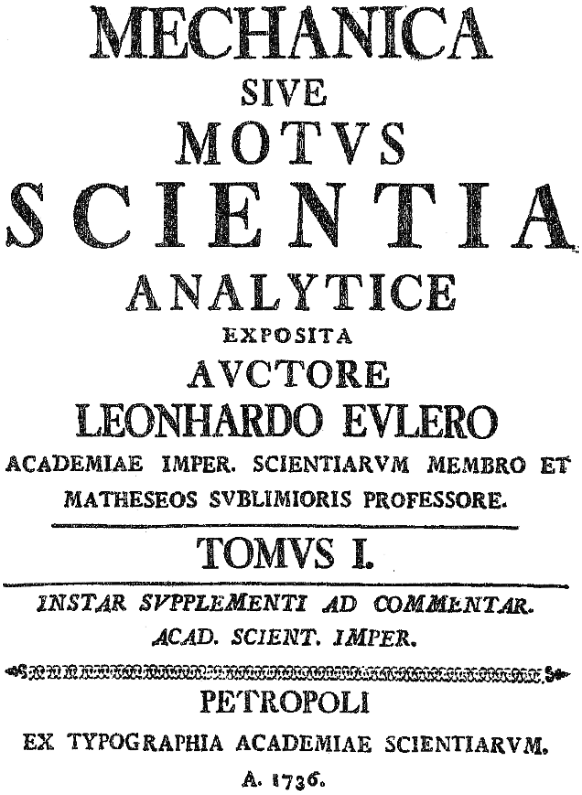
[47] Following Schrödinger who analyzed the "situation in quantum mechanics" in 1935 [Schrödinger, Naturwissenschaften], Euler introduced a "model" of the body and, following Archimedes, simultaneously a model of the world. The "world" consists of the lever and the vacuum. The observer is not a part of the world, but becomes a part if he replaces one of the bodies and is acting instead of this body ("Give me a place to stand" (compare Chap. 1)). Hence, from Archimedes' procedure it follows that the "observer" is also always an "actor" who may become a part of the mechanical system.

[48] This part of mechanics had been later called kinematics.

[49] The relativistic approach based on the criticism of Newton's concept of absolute motion was rediscovered by Mach in the 19th century [Mach, Mechanik] (compare Chap. 6).

[50] A new development had been only iniated by the invention of string theory in the 20th century where the "point" had been replaced with a "line" or "string. [http://www.superstringtheory.com/basics/index.html] "Ursprünglich war die Entdeckung der Strings (als 'duale Modelle') eine Formel von Gabriele Venenziano 1968 im Rahmen der Streumatrix-Theorie stark wechselwirkender Teilchen. 1970 gaben Nambu, Nielsen und SussKind eine Interpretation in Form von eindimensionalen Strings." [http://de.wikipedia.org/wiki/Stringtheorie]

[51] Later in the 19th century, this notion of arithmetization had been used by Felix Klein to describe the development of mathematics in the 19th century [Klein, Arithmetization].



MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.

TOMVS I.

INSTAR SVPPLEMENTI AD COMMENTAR.
ACAD. SCIENT. IMPER.

PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

Fig. 4.1 Front page of Euler's treatise *Mechanics or the science of motion analytically demonstrated* published in 1736. [Euler E015/016]

continued and crowned by Lagrange. In 1748, in the Prefaces of the *Institutiones*, Euler stated that he did not need any figures

> Hic autem omnia ita intra Analyseos purae limites continentur, ut ne ulla quidam figura opus fuerit, ad omnia huius calculi praecepta explicanda. [Euler E212, Preface]

and, 40 years later, Lagrange emphasized:

> One will not find figures in this work. The methods that I expound require neither constructions, nor geometrical or mechanical arguments, but only algebraic operations, subject to a regular and uniform course. [Lagrange, Mécanique]

Although Newton himself introduced the means to diminish the importance of geometry by the introduction of the calculus, he passionately and emphatically defended the geometrical foundation (compare Chaps. 2 and 3). Hence, Newton invented a powerful algorithm for the representation of rest and motion in terms of time and space, i.e. temporal and spatial variables and functions dependeing on these variables, but presented his discoveries in the traditional language of geometry. In the 17th century, the prototype of "arithmetization" is represented by the "indivisibles" which had been introduced by Cavalieri [Cavalieri] being different from continuous quantities. Continuous quantities can be divided in infinity (compare the analysis of Euler [Euler E842, Chap. 2]) whereas the process of division is terminated by the assumption of indivisibles. As a precaution, Newton excluded also painstakingly the complementary process, the generation of a line by the "apposition of parts" [Newton (Collins), Commercium] (compare Chaps. 2 and 3). Beside the relations between mechanics (physics) and geometry (mathematics) discussed in the frame of the continuum and the indivisibles, there are the logical, metaphysical and theological implications and representations which prepare the background for the former relation.

Euler continued the work of Cavalieri.[52] Euler did not reduce the dimension (extension) of the geometrical objects, but reduced the magnitude (extension) of one and the same object by dividing it into smaller parts, e.g. by dividing an extended line into parts being also lines, but smaller in magnitude. The magnitude is expressed arithmetically by the *division of a number* representing the length of the whole line into equal ratios where these numbers represent the length of the parts of the line (performing the division as an arithmetical procedures, the resulting parts can be interpreted geometrically as the "parts of the line the line consists of"). Hence, independently of the magnitude of the parts, it is always possible to assign a "length" different from zero to the parts provided that the length of the undivided line, i.e. the line in its "initial state" being a whole (and, following Leibniz, consisting of indeterminate parts [Leibniz, Fragmente]), is different from zero. Mediated by the *arithmetical* procedure, the "*indeterminate* parts" had been transformed into "*determinate* parts". Then, the "length of a part" can be *assigned* to each of the parts provided that the arithmetical operation is well-defined.

Newton combined the exclusion of indivisibles with the acceptance of atoms whereas Euler excluded the existence of atoms as a consequence of the properties of body to belong to the extended things [Euler E081]. An extended thing can be divided in infinity. Hence, it is not only impossible to find inextended least particles, but least particles at all. Euler replaced the geometrical model with an arithmetical

[52] "Young Cavalieri joins the not particularly famous and barely popular order of Jesuits. In Pisa's Jesuite monastery, where he lives, he is initiated in Mathematics by the Benedictian monk and student of Galileo, Benedetto Castelli (1577–1644), professor of Laurent of Medici. In 1629, with the support of Galileo, he was elected professor of mathematics at the University of Bologna, seat that he had until his death. Cavalieri's name is associated with the indivisible theory, which he has officially formulated in his book 'Geometry of Indivisible' in 1635. This theory could be considered as the beginning, which lead to the creation of infinitesimal calculus by Newton and Leibniz. Cavalieri in his book Geometria indivisilibus considered ≪all the lines≫ (Omnes lineae) as a basic notion he tries to calculate all the surfaces and volumes by dividing the areas in lines and the volumes in levels, that is he considers that a geometrical object of two or three dimensions is consisted by geometrical objects with dimensions smaller than two and three relatively. These elements, which are equally parallel line segments for the areas and equally parallel levels for the volumes, are mentioned as indivisibles of area or volume relatively. In 17th century Cavalieri is an isolated point, since he is the only one who tries to expand Eudoxus' magnitude theory in quantities with infinite points, without bothering for the composition of the continuous, uniting this way the new era with ancient Greek tradition."
[http://www.hms.gr/eme/modules/wfsection/article.php?articleid= 1074&lang=english]

model in a two step procedure. In the first step, Euler represented the curvilinear motion in terms of two or three straight motions and called the procedure the "art of resolution" ("Auflösungskunst") [Euler E842] which is simultaneously an "art of composition". In the second step, the elementary translations along the straight lines are represented in terms of *finite* and *infinitesimal* translations.

Euler removed the problem by resolving the curvilinear motion in a plane (in space) into two straight motions in two directions oriented perpendicular to each another. Therfore, the geometrical relation between a *polygon* and a *curved* line is replaced with the geometry of *one and the same* straight line.

> 33. Quemadmodum enim in geometria curvarum linearum elementa ut lineolae recta considerantur, ita etiam simili modo in mechanica motus inaequabilis in infinitos aequabilis resolvitur. Vel enim severa elementa aequabili motu percurruntur, vel mutatio celeritatis per huiusmodi elementa est tantilla, ut incrementum aut decrementum sine errore negligi possit.
> 34. Omnis ergo celeritatis mutatio in motu inaequabili in singulorum elementorum initiis fieri concipienda est, qui integra elementa aequabili motu percorri ponuntur.
> 35. Quare secundum notandi modum analyseos infinite parvorum, si celeritatis in primo elemento fuerit c, erit celritatis in secondo $c + dc$, in tertio $c + 2dc + ddc$, et ita porro.
> 36. Demonstrationis datae vis hoc nititur fundamento, quo celeritatis mutatio, quae fieri potest, dum elementum infinite parvum percurritur, debeat esse infinite exigua et evanescere prae celeritate, quam corpus iam habet. [Euler E015/016, §§33–36][53]

Translation and motion are described in terms of *finite* or *infinitesimal* increments of the path. The mechanical relations are assumed to be the same in different models, i.e. the finite velocity is defined by the geometric ratio of either finite or infinitesimal spatial and temporal intervals, i.e. either $v = \Delta s/\Delta t$ or $v = ds/dt$, respectively. The relation between Δs and ds is determinate by the rule that the latter is incomparable less than the former. Arithmetically, the result is expressed by the relation $\Delta s \pm ds = \Delta s$. This rule for the *arithmetical* ratio between a finite and an infinitesimal quantity had been formulated by Bernoulli in 1691–1692 [Bernoulli, 1691–1692].[54] Using Leibniz's terminology, the infinitesimal increment can be denoted as *syncategorematic* with respect to a finite velocity ("debeat esse infinite exigua et evanescere prae celeritate, quam corpus iam habet" [Euler E015/016, §§ 36 and 121], compare also [Arthur, Fictions]).

Based on the analysis of motion and the change of motion due to the presence of forces [Newton, Principia, Second Law], Euler developed a model where (i) finite and (ii) infinitesimal quantities had to be necessarily included because of the two different mechanical states of the body described by rest and motion [Newton, Principia, First Law]. Therefore, three quantities had to be distinguished which are related to the three possible types of motion (i) rest, (ii) uniform motion and (iii) non-uniform motion, described by (i′) $v = 0$, (ii′) $v \neq 0$, $v = \text{const}$ and (iii′) $dv \neq 0$

[53] "The strength of the demonstration is due to the fact that the possible change of the velocity, which is able to happen while an infinitely small element is run through, is itself infinitely small and vanish with respect to the velocity with which the body already is moving." [Euler E015/016, § 36]

[54] Quantities whose addition or subtraction does not change a given quantity are called ciphers. Euler made use of this denotation [Euler E212, §§ 75–90] (compare Chap. 5)

(change of rest) and $v \pm dv,\ dv \neq 0$, (change of motion), respectively. The change of rest and motion is described by an *infinitesimal* quantity currently mathematically represented in the frame of nonstandard analysis and denoted as $dv \approx 0$. Then, the only real infinitesimal number zero can be distinguished from other infinitesimal quantities being different from zero [Keisler] (compare Chap. 5).

Following Euler, the common rule for the change of rest and motion can be only mathematically formulated if those quantities as the *infinitesimal* increment (decrement) of velocity dv had been assumed. The evidence obtained from mechanics is not at all a rigorous and strong mathematical demonstration that these quantities are of a sound mathematical origin, but an essential hint that they may be proved to have a right of existence in the world of numbers. The problem to define numbers having such properties had been only solved in the 20th century [Keisler]. Not suprisingly, that scholars of the 18th century are not enthusiastic in handling of such objects and were seeking for alternatives [d'Alembert, Encyclopédie, Limit], [Lagrange, Fonctions].

From the analysis of Kästner's *Anfangsgründe* we obtain an interesting result[55]: (i) In the first half of the 18th century, the theories of Newton and Leibniz are simultaneously reconsidered and recognized, (ii) after 1740, Euler's *Mechanica* is used as a summary of the state of art and as a basis of the further development, (iii) Newton, Leibniz and Euler are only partially acknowledged and accepted,[56] (iv) new concepts had been introduced by d'Alembert and Maupertuis, (v) Euler created a new consistent and almost complete representation of mechanics based on a unification and merging of Newton's and Leibniz's legacy.

Although Kästner studied carefully Euler's treatise, he could not reproduce the full content of Euler's *Mechanica* especially the innovations as far as the relations between inertia and forces are concerned. However, it should be mentioned that Euler explicitly elaborated in more detail the hidden subtext of *Mechanica* only some years later in the *Anleitung* [Euler E842, Chaps. 2, 3, 4, 5 and 6][57] and the *Theoria* [Euler E289]. However, the difference between the papers resulting from the reception of Euler's ideas by the contemporaries and Euler's own later contributions demonstrates the difficulties and obstacles which were caused by the persisting traditional schemes. Nevertheless, at the same time, the progress Euler made in com-

[55] The same conclusion is obtained from the analysis of Châtelet's Institutions [Châtelet, Institutions].

[56] Compare d'Alembert's comment on Euler's *Mechanica* [d'Alembert, Traité].

[57] Before 1750, Euler stated: "Mit dem Worte Trägheit ist man auch gewohnt, eine Kraft zu verbinden und dem Körper die Kraft der Trägheit zuzuschreiben, wodurch grosse Verwirrungen veranlasset werden; denn da eine Kraft eigentlich dasjenige genannt wird, welches vermögend ist, den Zustand eines Körpers zu verändern, so kann dasjenige, worauf sich die Erhaltung eben desselben Zustandes gründet, unmöglich als eine Kraft angesehen werden. Wird nun anstatt dieses verführerischen Worts ein anders, so die Beschaffenheit der Sache genauer ausdrückt, in Gebrauch gebracht, so werden alle dergleichen Verwirrungen vermieden." [Euler E842, § 31] In 1765, Kästner stated: "51. Lehrsatz. Jede Kraft strebt Bewegungen hervorzubringen. (...) Eben das, daß die Trägheit nie strebt Bewegungen hervorzubringen, macht es bedenklich, ihr den Namen Kraft beyzulegen. Will man solches tun, so macht sie allein eine ganz besondere Gattung von Kräften aus, von der alle übrigen verschieden sind." [Kästner, Anfangsgründe. I (51)]

parison to other scholars of the 18th century is higlighted. Paul Stäckel, the editor of the *Mechancica*, summarized that Euler invented a "completely new approach of the theoretical representation of mechanics being also different from Newton's axiomatics in the *Principia*" [Euler, Opera Omnia, II, 1, Preface (Stäckel)].

4.1.3 Rest and Motion: Internal Principles

Bodies at rest and in equilibrium had been studied by the ancients [Newton, Principia], [Leibniz, Specimen], [Euler E015/016]. The completion and generalization is the theory of motion or the science of motion ("mechanica sive motus scientia") which may be properly named *dynamics* to accentuate the difference and simultaneously the connection to *statics* [Bernoulli, Letter to Euler 1737].[58] However, following Euler, his program may be interpreted as the unification of statics and dynamics since rest and motion are always treated as correlated notions. The reason is that Euler regarded rest and motion always as *relative* rest and motion (compare Chap. 6) and both notions cannot be defined independently, but only as correlated to each another. As a consequence, Euler introduced rest and motion as basic concepts already in the first paragraph of the chapter entitled *De motu in genere*. The objects being described are bodies which are different from empty space.[59] The empty space as a basic concept[60] had been explicitly discussed in the treatise *Anleitung* [Euler E842, Chaps. 2, 3, 4 and 5].

The properties of bodies are related to rest and motion since the properties are not only defined by the common properties the bodies have with space, but mainly by the difference between space and bodies. The space is assumed to be immobile. Hence, the bodies are distinguished from space, the immobile object, by their *mobility*. The operation for mobile objects is defined as a *translation*. The difference between the possible types of translations is not considered.

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[58] "Der Titel *Mechanica* sei nicht passend, weil damit von alters her die Lehre von den 'toten' Kräften (Statik) bezeichnet werde; nach dem Vorgang von Leibniz solle man die Wissenschaft von den 'lebendigen' Kräften Dynamik nennen." [Euler, Opera Omnia, II, 1, Preface (Stäckel)] Here, Johann Bernoulli proposed to preserve the division into "statics" and "dynamics" as two different and independent parts of the theory of bodies whereas, according to Euler's plan, there is no difference between statics and dynamics as far as the relations between bodies and forces is concerned. Euler stated [Euler E015/026, § 213], and this theorem had been renewed by Kästner [Kästner, Anfangsgründe, § 52] who referred to Euler, that it is the same force which causes either the pressure or the motion. "Man ersieht hieraus, dass jede Kraft eine doppelte Wirkung auf die Körper ausübt; die eine, wodurch sie ihnen ein gewisses Bestreben, sich zu bewegen, mittheilt und die andere wodurch diese wirklich zur Bewegung gelangen. Jene wird vorzugsweise in der Statik betrachtet und muss durch das (...) Gewicht gemessen werden. (...) Die andere Wirkung muss aber durch die Beschleunigung oder die Zunahme der Geschwindigkeit (...) gemessen werden." [Euler E015/016, § 213] (the difference is traced back to different measurements of the same quantity, in the present case, the gravitational force).

[59] For the difference see Sect. 4.2.1 on *extension* and *mobility*.

[60] In an early version of the theory of motion, Leibniz also based mechanics on the difference between bodies and empty space (vacuum) [Leibniz, Hypothesis].

The two possible "states" are first the translation and second the absence of a translation. Euler did not consider how the translation is really performed and did not discuss the causes for the translation from one place into another place, but accentuated that the *occupation of the place* is an essential property of all bodies. Moreover, "the idea of rest and motion is only valid for things which can occupy a place". The place is a part of the immobile space. Different parts of the space can be only distinguished by the presence and absence of bodies. The ability to occupy a place is the property which distinguishes the bodies from other things.[61]

Following Newton, Euler assumed rest and motion as the states of bodies which are described geometrically and mechanically, i.e. by the absolute or relative positions and by the *preservation* or *changes* of theses positions. Every body is either moving or resting. Hence, a body which is neither moving nor resting cannot exist.[62]

> 1. Motus est translatio corporis ex loco, quem occubabat, in alium. Quies vero est permansio corporis in eodem loco.[63]
> 2. Motus igitur et quietis idea in alias res cadere non possunt, nisi quae locum occupant. Quare cum hoc dit corporum proprium, locum occupare, de solo corpore dici potest, quod moveatur vel quiescat.[64]
> 3. Nullum enim existere potest corpus, quod non vel moveatur vel quiescat.[65]

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[61] Compare the discussion of the relation between the concepts of extension, mobility, steadfastness and impentrabily Euler gave later in the *Anleitung zur Naturlehre* [Euler E842].

[62] Hence, every body is either moving or resting. Nevertheless, there may be such things which are *neither* moving *nor* resting, but these things do not belong to the genus of bodies [Euler E842, §§ 1–8]. Following Descartes who applied the same procedure in stating that "res extensa is neither res infinita nor res cogitans", Euler established a correlation between things belonging to different genera whose reliability is guaranteed by logically defined relations. Therefore, although any influence by ghosts had been definitively excluded ("49. Hier werden diejenigen Veränderungen mit Fleiss ausgeschlossen, welche unmittelbar von Gott oder einem Geiste hervorgebracht werden." [Euler E842, § 49]), the theory of bodies simultaneously requires an understanding of the nature of ghosts [Euler E344, Lettre LXXXV].

[63] Euler introduced in the following paragraphs the *absolute* and the *relative* position [Euler E015/016, §§ 4 and 7], explicated the model of absolute space [§ 8] and the relative motion and the relative rest [§§ 9–12].

[64] "Likewise, as rest is the permanent stay at the same place, so motion is the permanent change of the place. It is said, of course, that a body is resting which is always observed to stick at the same place; whereas a body is said to be moved which is successively translated into other and other places with the change of time." In the *Theoria*, Euler inverted the order of "rest" and "motion" to accentuate the role of *inertia* also in the case of rest which determinates the *state* of the resting body in the same manner as the states of the moving body. "1. Quemadmodum *Quies* est perpetua in eodem loco permanentia, ita *Motus* est continua loci mutatio." [Euler E289, § 1]

[65] The correlation between rest and motion had been also later accentuated as a fundamental guiding principle for the investigation of the "laws of motion" [Euler E197] (compare Chap. 7). "§ XII. (...) Ainsi l'effort, ou la somme des efforts, étant pour un instant quelconque de mouvement $= \Phi$, & posant l'élément du tems $= dt$, il faut que cette formule intégrale $\int \Phi dt$ soit un *minimum*. De sorte que si pour le cas de l'équilibre la quantité Φ doit être un *minimum*, les mêmes loix de la Nature semblent exiger, que pour le mouvement cette formule $\int \Phi dt$ soit la plus petite." [Euler E197, § XII] Later, the relation between rest and motion had been drastically changed and finally inverted. Eventually, rest played a subsidiary role as a "limiting" or special case of motion. (This

> 13. Omne corpus, quod sive motu absoluto sive relativo in alium locum transfertur, per omnia loca media transit neque subito ex primo in ultimum potest pervenire.
> 14. Sequitur ex his etiam motum non posse fieri in instanti, sed tempore opus esse, quo ex alio loco in alium perveniat corpus. Quia enim per singola loca media debet transire, hoc cum motu in istantaneo consistere non potest. [Euler E015/016, §§ 1–3, 13 and 14]

Usually, it is assumed that motion is a special type of translation and rest is a special case of motion (Leibniz: rest is an evanescent motion). The conclusion is that the bodies are necessarily either moving or resting. Newton did not start with rest and motion, but with the relation between space and place. The place is a part of the space which is filled by the things. The place is completely filled by the body.[66] Instead of the definition "Motion is the change of the place", Euler stated "Motion is the translation of the body from the place it ocupies into another place". The *occupation of place* is only possible for those things which are either moving or resting. Therefore, the *common* principle valid for every body is the *space occupation* which is not specific either for rest or for motion,[67] but is fulfilled independently of the state of the body and of the change of the state. Already in Def. 5 Newton introduced the notion of force which is assumed to be "the causal principle of motion and rest". The advantage of this construction is that a *unified theory* of *rest and motion* is introduced based on the concept of "force" (vis) forming the basis for the unification of the theory of *equilibirium* (constructed by Archimedes and other scholars) and the theory of *motion* (which Newton aimed to invent). The same problem has to be solved by Galileo, Descartes, Leibniz and their followers since the theory of equilibrium constructed by the ancients was the indispensable fundament of the theory of motion. In the 20th century, the problem to extend and generalize a theory was recovered after the discovery of the action parameter by Planck. The

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interpretation was also promoted by Leibniz who assumed that rest is an "evanescent motion" and that the "rules for resting bodies follow from the rules for moving bodies". "Unde consequens est Leges motuum tales assignari debere, ut non sit opus peculiaribus regulis pro corporibus aequalium et quiescentibus, sed hae ex regulis corporum aequalium et motorum per se nascantur, (...)." [Leibniz, Specimen, II (4)]) Obviously, Euler inverted the Leibnizian procedure and obtained the "rules for motion" by a *transfer* of the basic principles from the "rules for rest" to the case of motion. Hence, Leibniz's interpretation and application of principle of continuity hampered the analytical representation of motion since rest and motion are not treated on an equal footing using a unified approach for the invention of principles like the principle of least action (compare Chap. 7)

[66] "Def: 1. Locus est spatii pars quam res adæquate implet. Def: 2. Corpus est id quod locum implet. Def: 3. Quies est in eodem loco permansio. Def: 4. Motus est loci mutatio. Def: 5. Vis est motûs et quietis causale principium. Estque vel externum quod in aliquod corpus impressum motum ejus vel generat vel destruit, vel aliquo saltem modo mutat; vel est internum principium quo motus vel quies corpori indita conservatur, et quodlibet ens in suo statu perseverare conatur & impeditum reluctatur." [Newton, De gravitatione] Newton distinguished between *external* and *internal* principles being related to *different kinds* of forces. Leibniz's mechanics is also based on the distinction between different kinds of forces, i.e. (i) dead and living forces (corresponding to the theory of equilibrium of the ancients and the theory of motion which has to be developed), (ii) active and passive forces and (iii) primitive and derivative forces [Leibniz, Specimen] (compare Chaps. 2 and 3).

[67] The same assumption (related to the axiomatic frame) is made for the inertia or steadfastness which is assumed to be independent of the rest, motion and, additionally, of interaction of bodies (compare Sect. 4.5).

classical mechanics was the indispensable fundament of the new theory, but basic principles of classical mechanics were in contradiction to the experimental findings and theoretical models developed for their explanation.[68]

The radical turn-over Euler made was to *preserve* the distinction between "internal" and "external" principles, but to *replace* the interpretation of rest and motion in terms of *forces* with the interpretation in terms of *states* (which are not related to forces) and of the *change of the states* (which are due to forces), respecticely.

Euler developed a new concept which was also directly represented by the table of contents of the *Mechanica*. The first and second chapters are entitled "De motu in genere" and "De effectu potentiarum in punctum liberum agentium", respectively. Although Euler discussed the motion of a body, the notion of force is explicitly introduced only in Chap. II in § 99. In Chap. I, the forces are only implicitly discussed being related to the distinction between uniform and nonuniform motion [Euler E015/016, § 13–32, §§ 33–37].

Thirty years later, Euler chose the same title for the first chapter of the *Theoria motus corporum solidorum seu rigidorum* and the first paragraph are preserved, however, the order of rest and motion is interchanged. From the table of contents, the distinction between internal and external principle is seen to be the basic principle of the analytical demonstration of the theory of motion.[69]

In both books, the *Mechanica* (1734–1736) and the *Theoria* (1765), rest and motion are clearly distinguished, but related to each other by the same general principles.

> 1. Quemadmodum *Quies* est perpetua in eodem loco permanentia, ita *Motus* est continua loci mutatio. *Corpus scilicet, quod semper in eodem loco haerere observatur, quiescere dicitur: quod autem labente tempore in alia atque alia loca succedit, id moveri dicitur.* [Euler E289, § 1]

According to the distinction between internal and external principles and, moreover, in contrast to Newton and Leibniz, the forces are explicitly excluded from the internal properties of bodies (compare Table 4.1). The only *necessary* and *sufficient* internal property is the impenetrability [Euler E842, Chaps. 1, 2, 3, 4, 5 and 6].[70]

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[68] "Eine physikalische Theorie glauben wir dann anschaulich zu verstehen, wenn wir uns in allen Fällen experimentelle Konsequenzen dieser Theorie qualitativ denken können, und wenn wir gleichzeitig erkannt haben, daß die Anwendung dieser Theorie niemals innere Widersprüche enthält." [Heisenberg, Anschaulich] The theory should be free of contradictions not only the application, but also the construction. Euler demonstrated the internal contradictions of the Leibniz-Wolffian theory of "simple things". The internal contradictions being inherent in a theory had been later called "antinomies" by Kant.

[69] "Cap. I. Consideratio motus in genere, Cap. II. De internis motus principiis, Cap. III. De causis motus externis seu viribus, Cap. IV. De mensuris absolutis ex lapsu gravium petitis, Cap. V. De motu absoluto corpusculorum a viribus quibuscunque actorum, Cap. VI. De motu respectivo corpusculorum, a viribus quibuscunque sollicitatorum." [Euler E289]

[70] "Capitel 5. Von der Undurchdringlichkeit als der vierten allgemeinen Eigenschaft und dem Wesen der Körper. (...) § 38. Die Undurchdringlichkeit schliesset für sich schon die Ausdehnung und Beweglichkeit und folglich auch die Standhaftigkeit in sich." [Euler E842, Chap. 5] The impenetrability characterizes the nature of bodies (das Wesen der Körper) and makes the diffeence to other thongs, e.g. ghosts.

Table 4.1 Internal and external principles in Newton, Leibniz and Euler

.	Internal	External
Newton	Force (of inertia)	Force (impressed moving)
Leibniz	Force (primitive)	Force (derivative) due to interaction
Euler	No force (of inertia)	Force (due to interaction)

> 75. *Interna motus principia* complectuntur omnia ea, quae in ipsis corpora insunt, in quibus ratio sive quietis sive motus eorum contineatur exclusis omnibus causis externis, quae quicquam ad eorum motum quietem conferre queant. [Euler E289, § 75][71]

The carefully formulated and logically and syntactically correct representation of the basic principles is the indispensable fundament of the introduction, validity and applicability of an algorithm defined for the quantities appearing in the mechanical model. All theorems for rest and the change of rest are *transferred* to the case of uniform motion and the change of uniform motion. Hence, remembering that the science of equilibrium (or rest) had been invented by the ancients, Euler presented a generally valid representation for statics and mechanics (or dynamics). The basic principles known from statics are transferred to dynamics as Euler demonstrated for the methods of Archimedes [Euler E015/016, § 56,[72] § 107]. The transfer is only justified and can be only completely performed if the *analytical* formulation of theorems can be related to different semantic interpretations, e.g. either for rest or for motion. The direct way is to demonstrate the common syntactical structures of the theorems related to rest and motion.[73] Then, also the mathematical representations are properly defined by a common algorithm. There are two cases to transfer the algorithm, first, to transfer the algorithm defined for the change of rest to the change of motion and, second, to transfer the algorithm defined for the change of motion to the change of rest (see below).

In the textbooks *Mechanica* and *Theoria*, the demonstrations are dominated by the logically consistent application of basic principles and theorems and the deductive development of the consequences. The polemic and criticism of previous and contemporary theories is only fragmentarily included, but comprehensively elaborated in the *Anleitung zur Naturlehre* [Euler E842] and later in the *Lettres à une*

[71] "The *internal principles of motion* embrace all that which is inherent in the bodies themselves, and in which is retained the cause of either their rest or motion if all external causes are excluded which contribute to rest or motion of them." [Euler E289, § 75]

[72] There is a transfer of the rule valid for bodies in empty space into the real world. Archimedes demonstrated the law of the lever in empty space and in real world.

[73] "Or on énonce communément ce principe pat deux propositions, dont l'une porte, *qu'un corps étant une fois en repos demeure éternellement en repos, à moins qu'il ne soit mis en mouvement par quelque cause externe ou étrangere*. L'autre proposition porte *qu'un corps étant une fois en mouvement, conservera toujours éternellement ce mouvement avec la même direction et la même vitesse, ou bien sera porté d'un mouvement uniforme suivant une ligne droite, à moins qu'il ne soit troublé par quelque cause externe ou étrangere*. C'est en ces deux propositions que consiste le fondement de toute la science du Mouvement, qu'on nomme la Mécanique." [Euler E343, Lettre LXXIII]

princesse d'Allemagne [Euler E343], [Euler E344], [Euler E417].[74] In the *Lettres*, Euler explicitly commented on problems which had been in earlier writing only cautiously, shortly and implicitly discussed, e.g. the rejection of the "force of inertia". Although Euler implicitly rejected the "force of inertia" twenty years ago in the *Mechanica*, he also commented on this topic thirty years later since it seems to be necessary to repeat the earlier argumentation [Euler, Anleitung][75] in 1765.

> 92. Proprietas illa corporum, quae rationem perseverationis in eodem statu in se continet, *inertia* appellatur, quandoque etiam *vis inertiae*.
> 95. Vox *inertia* proprie ad eam corporum proprietatem indicandum, qua quiescentia in quiete persistunt, est adhibita, propterea quod in hoc statu motui se quasi opponunti sed quia corpora, in motu constituta, aeque se omni mutationi ratione, tam celeritatis quam direction is, opponunt, hoc nomen haud inepte ad conservationem status, sive quietis sive motus indicandam usurpatur. Vocatur etiam passim *vis inertiae*, quia *vis* est aliquid mutationi status reluctans; sed si vis definitur per causam quamcunque, qua status corporum mutatur, hic in ista significatione neutiquam accipi potest; eius certe ratio maxime discrepat ab ea, qua deinceps vires agere ostendemus. Quare, ne hinc ulla confusio oriatur, nomen *vis* omittamus et hanc corporum proprietatem simpliciter nomine *inertia* appellabimus. [Euler E289, §§ 92 and 95][76]

This discussion and clarification indicates an essential step in the development of the post-Newtonian and post-Leibnizian mechanics starting from the 17th century frame of the relations between *bodies and forces* established by Newton and Leibniz[77] to the 18th century theory which is in main features identical with the currently accepted version. Euler rejected all kinds of forces which had been assumed by Newton and Leibniz, first of all those which had been called *inherent* or *primitive* forces. The only type of forces which survived the procedure is the Newtonian

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[74] Although the *Lettres* are often considered as a popular book, the criticism of different schools in the 17th and 18th centuries is precise, excellent and almost complete. Furthermore, Euler commented on his own theory and discussed the relations to Newton, Leibniz, Wolff and Maupertuis.

[75] Certainly, we have to admit that the *Anleitung* had not been published, but Euler argued also earlier (in 1750) in favour of the name *inertia* instead of *force of inertia* because of the misleading physical interpretation [Euler E181, §§ 5–10].

[76] "The word *inertia* will be especially applied for the indication of such properties of bodies in virtue of the resting bodies persist at rest because they quasi resist to motion in this state of rest, but since the bodies being in motion resist also each change of their velocity and direction the name is not improperly used for the conservation of state, either the state of rest or the state of motion. Sometimes one says force of inertia because force is somewhat which counteracts the change of the state, but if the force is defined by an arbitrary cause which changes the state of the body then it can be by no means supposed in that meaning. Its properties differ in the highest degree from that manner the forces are acting as we will show. So that no confusion may arise, the word *force* will be omitted and we will simply call this property of bodies by the name *inertia*." [Euler E289, § 95]

[77] Usually, the essential differences between Newton and Leibniz had been accentuated, e.g. as far as the theory of time and space is concerned (compare Chap. 2). However, in the beginning of the 18th century, both scholars had been experienced as almost equivalent which was convincingly demonstrated in the debate on the priority in the invention of the calculus (Equivalence and Priority [Meli]). Later in the 19th and 20th centuries, as people were completely accoustomed to Newtonian mechanics they were not aware of the fact that they are not using the Newtonian, but the Leibnizian language of the calculus to formulate and to solve mechanical problems by reckoning.

"impressed moving force" and its modified version being related to the Leibnizian "derivative forces" resulting from the impact of bodies.[78]

Euler distinguished between internal and external states of bodies where the internal state is desribed in terms of the relations between different parts of the bodies whereas the external state is obtained from the relation of the whole body (including its internal parts) to other bodies.[79] The definition of the states of rest or uniform motion is based on the external relations to other bodies and, additionally, on the relations between the body and the space [Euler E149]. Euler claimed that the preservation of direction of a uniformly moving body cannot be explained by Leibniz's relational theory of space and time [Euler E149, §§ 20 and 21].[80] Therefore, Euler accepted absolute time and space, but rejected absolute motion (compare Chap. 6).

4.1.4 From Geometrical to Analytical Representation of Mechanics

Euler generalized the commonly used models for the relations between straight and curved lines. The most prominent model had been established between the circle and the inscribed and circumscribed polygons which had been used before and after Euler by Newton [Newton, Principia] (compare Chap. 2) and Lagrange [Lagrange, Fonctions] (compare Chap. 5) and almost all other scholars. Geometrically, this model represents a special relation between straight and curved lines in a plane

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[78] The impressed moving force is assumed to be "impressed upon a body" whereas the "derivative forces" are assumed to be generated by the bodies. The essential difference is that an isolated body is unable to change its state ("quantum in se est" [Descartes, Principia]) due to the lack of these forces. Nevertheless, in almost all 19th century and present textbooks the introduction into *Mechanics* is based on the *one body – one force* model describing the dynamics of a material point (compare e.g. Helmholtz [Helmholtz, Vorlesungen, Erster Theil, Dynamik eines materiellen Punktes]). Helmholtz based the construction and justification of the basic law on the need for causality ("Causalitätsbedürfnis") which can be clearly related to the Kantian foundation of knowledge [Kant, KrdrV].

[79] "30. *Man sagt, ein Körper verbleibe in ebendemselben Zustande, wenn derselbe entweder in Ruhe verbleibt oder seine Bewegung nach ebenderselben Richtungmit mit einerlei Geschwindigkeit fortsetzet.*
Man kann sich in einem Körper einen doppelten Zustand vorstellen, den äusserlichen und den innerlichen. Dieser bestehet in der Art der Theile, aus welchen der Körper bestehet, und ihrer Zusammensetzung selbst; der äusserliche Zustand aber, von welchem allhier allein die Rede ist, besteht in den Verhältnissen des Körpers mit dem Raume. So lange sich nun ein Körper in Ruhe befindet, so bleibt er an ebendemselben Ort und ist also kein Zweifel, dass er nicht in ebendemselben Verhältnisse mit dem Raume verharren sollte." [Euler E842, § 30]

[80] "21. The question here is not our estimation of the equality of time, which will no doubt depends on the state of our mind; but the question is the equality of time during which a body put in a uniform movement runs equal spaces. Since that equality cannot be explicated in terms of the order of successions, no more than the equality of space in terms of the order of coexistence, and that equality is essentially involved in the principle of movement; one cannot say that the bodies, in continuing their movement, depend on a thing which does not exist but in our imagination." [Euler E149, § 21 (Uchii)]

and, mechanically, motions confined to a plane[81] like the ancient model of motion of planets on circles, Kepler's planetary motion on ellipses, Galileo's analysis of the path for composite motion on a parabola and Newton's model of planetary motion. In the latter case, although the force is acting in the whole space, the 3D problem is also reduced to a 2D problem (compare Chap. 2). Hence, Johann Bernoulli commented that the solution of the direct problem is missing where, in modern terminology and already assumed by Newton, the attraction or the gravitation field is not confined to the plane the body is moving, but is extended in the whole 3D space. Only in this case, the force between two distant bodies is described by $1/r^2$ dependence on their distance r. "In 1710, Jean Bernoulli pointed out that Newton had not proved Kepler's law of ellipses but only its converse (...)" [Timeline, Park].[82] Thirty years later, in 1742, Johann Bernoulli in a paper published in *Opera omnia*, proved that the orbits of objects bound by the inverse square force are conic sections [Timeline].[83] Euler's treatise on this type of motion is entitled *Methodus inveniendi lineas curves maximi minimve proprietate gaudentes sive solution problematis isoperimetrici lattissimo sensu accepti* and devoted to the isoperimetric problem [Euler E065] being simultaneously a generalization of Galileo's and Newton's analysis of the motion in a plane.

The crucial problem was to relate the curved to the straigt and, vice versa, the straight to the curved. It is automatically reconsiderd and reformulated by the inclusion of the other two main kinds of motion, first the motion along a straight direction and second the motion performed neither along a straight direction nor bound to a plane. All three problems are formulated in the same frame of reference given by the extension of the space. Decomposing the *homogeneous* space into three independent directions represented by straight lines which are oriented

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[81] This procedure can be applied to planetray motion since the paths of the planets are confined to the plane which is fixed in space by the angular momenten (compare Chap. 2). Newton discussed central forces for the motion of a body along a path formed by an inscribed polygon [Westfall, Never, p. 149] (compare Fig. 2.1). The model follows from Kepler's 3rd Law. Replacing the circular motion by the motion along ellipsoidal paths of the planets Kepler stepped beyond the Greek tradition, but he preserved the Greek model for forces since the force is acting parallel to the tangent of the curvilinear path. However, instead of the $1/rr$ law for central forces, Kepler assumed an $1/r$ law for the force between the sun and the planets because the motion is confined to a plane [Simonyi, pp. 192–193].

[82] In the 19th and 20th centuries, Bertrand [Bertrand] and Ehrenfest [Ehrenfest 1917], [Ehrenfest 1920], [Ehrenfest] analyzed the *stability of trajectories in dependenc on the dimension of the spaces* the bodies are moving under the influence of central forces. The result is that the trajectories are stable only in 3D space which is geometrically distinguished from spaces of lower or higher dimensions by the relation between the number of straight lines and the number of planes. Geometrically, in 3D space there are three and only three straight lines and three and only three planes perpendicular to each other, mechanically, "the dualism between three components of the force and the three components of a pair of forces which together can replace an arbitrary system of forces. (...) Let the number of axes of coordinates be n, taking two of them at a time we can draw through them $P = n(n-1)/2$ planes." [Ehrenfest] Only for $n = 3,\ P = 3$ is also equal to three.

[83] Further famous 2D problems are: (i) Refraction of the light (Descartes), (ii) Maupertuis' principle applied to the problem of light refraction [Maupertuis, Accord]. (iii) In the application of the Max min method for the reflexion of light, Leibniz assumed also a 2D model [Leibniz, Hecht].

perpendicular to three independent planes [Euler E842] all types of motion can be simultaneously described and *analytically represented* by three time dependent functions $x = x(t),\ y = y(t)$ and $z = z(t)$ whose correlation is due to the motion of bodies. First, the 1D motion along a straight line say the x-direction, described by $x = x(t),\ y = y(t) = 0$ and $z = z(t) = 0$ (performed *perpendicular* to the y,z-plane), second, the 2D motion *confined* to the x,y plane, $x = x(t),\ y = y(t)$ and $z = z(t) = 0$ and, third the 3D motion which is neither along a straight direction nor confined to a plane, i.e. $x = x(t),\ y = y(t)$ and $z = z(t)$. Euler called this procedure "Auflösungskunst" (the art of resolution) [Euler, E842].[84]

Euler based the theory of any kind of motion *geometrically* on the motion along a straight line and *analytically* (arithmetically) on the representation of motion by scalar functions of one variable called time, $x = x(t)$, whose *inverse* function $t = t(x)$ is, consequently, nothing else than the *time*. Mathematically, Euler introduced not only the position of a body as a function *depending* on time, but also time t as a function *depending* on position or coordinates s by the relations (established between time, coordinates and velocity v) $ds = v \cdot dt$ and $dt = ds/v$, respectively [Euler E015/016, §§ 18–37].

The geometrical model of motion is completely replaced with an analytical model made up of time-dependent functions $x = x(t),\ y = y(t)$ and $z = z(t)$ and the velocities being either time independent or also time dependent functions $u = u(t),\ v = v(t)$ and $w = w(t)$ defined for motions along the directions defined by the x, y and z axis, respectively. The correlation between the geometrically defined directions is not due to the time, but due to the *motion* of the body. Hence, "time" becomes a "motion related" parameter,[85] especially the equality of time intervals for uniform motion [Euler E149, §§ 19–21] (compare Chap. 6).

Here, Euler introduced a one to one correspondence between coordinates and time, i.e. x and t, as a prototype of the relation between time and space which is subsequently transferred to the 2D and 3D motions by adding additional scalar time

[84] The "art of resolution" is simultaneously the counterparts of an "art of composition" summarized in Newton's 3rd Law and the Corollary I. "Lex. 3. Actioni contrariam semper & aequalem esse reactionem: sive corporum duorum actiones semper esse aequales et in partes contraria dirigi. Corol. I. Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere, quo latera separatis." [Newton, Principia] Although Newton explicitly accentuated that there are always *pairs of forces* in nature and a solitary force never appeared without its counterpart (complement), the theory of mechanics is presented in the first chapters of almost all textbooks as the "theory of a mass point a force is acting upon". Dynamics is developed a theory of a "mass point" [Helmholtz, Vorlesungen]. The world consists of space, time, one body and one force impressed upon the body (compare Section "Euler's world models"). The corresponding part of Euler's *Mechanica* is entitled "on the effect of forces upon a free point". Newton's guiding idea is to destroy the basis of Descartes' relativism (compare Chap. 2). Hence, Newton presented the basic law in terms of an *asymmetric body-force relation* (2nd Law) and formulated the axioms such that the relations of the chosen body to other bodies can be omitted.

[85] Euler's representation of time as a motion dependent parameter had been later recovered by Einstein [Einstein, Relativity]. In contrast to Euler who assumed that ds and dt are independent of velocity, Einstein introduced $ds(v)$ and $dt(v)$ as depending on velocity. Nevertheless, Euler's representation is consistent since the increment of velocity dv had been also assumed to be *independent* of velocity v which is not the case in Einstein's theory (compare Chap. 6).

dependent functions $y = y(t)$ and $z = z(t)$ whose combination describes all possible motions of a body.[86] The important mathematical and mechanical result is that the distance becomes a *function* of time and time becomes a *function* of the distance. Now, the relation between the *geometrical* and *analytical* representation of the mechanical quantities is clearly defined and time and space are treated on an equal footing [Euler E289, § 18].[87] The preference which had been introduced by Newton due to the universal time dependence of fluents and fluxions had been also removed and temporal and spatial variables are of the "same kind". Euler assumed that "les idées de l'espace et du tems ont presque toujours en le même sort" [Euler E149].[88] The mathematical representation of the relation becomes completely independent of the geometrical model (of motion) and is analytically given by a straightforward procedure based on the calculus.

$$x = x(t) \qquad dx \sim dt \tag{4.1}$$

Following Newton and Euler, the *principle of continuity* is established as generally valid for motion and, as a consequence, transcreation[89] is completely excluded. The conservation of state is explained by the invariant properties of the bodies which distinguish these things from all other things [Euler E842, Chap. 1]. In Leibniz's theory, although the body is moving, the transcreation is not determined by motion, but, on the contrary, the motion is determined by transcreation, i.e. the discretization is not generated by the body, but externally caused by superior principles and actions [Leibniz, Monadology, § 47]. Following Euler, the continuity is ensured by the properties and the actions of bodies themselves [Euler E842, § 49]. Euler's assumption of "internal principles" [Euler E015/016], [Euler E289][90] guarantees the

[86] Obviously, the procedure can be continued by the introduction of additional functions $u(t)$, $v(t)$, $w(t)$, etc. which have to be also geometrically interpreted if $x(t)$, $y(t)$ and $z(t)$ had been related to geometry. The asymmetry between *time* and *space* is caused by the different number of variables, i.e. by the decision to admit only *one time* variable, but *three space* variables. This choice cannot be justified by mathematical arguments since there is no preference between the time and the space variable in case of 1D motion.

[87] "18. Cum loci ideam definiverim, prout eam quidem sensuum iudicium suppeditat, idea loci nunc quoque temporis, quae in notione quietis ac motus implicatur, occurrit. (...) Tempus igitur perinde nobis liceat in calculum introducere, ac lineas aliasque magnitudines geometricas." [Euler E289, § 18] Following Euler, Lagrange claimed: "Ainsi, on peut regarder la Mécanique comme une Géométrie à quatre dimensions et l'Analyse mécanique comme une extension de l'Analyse géométrique." [Lagrange, Fonctions, Part III, § 1]

[88] "18. The ideas of space and of time have almost always been of the same sort, so that those who have denied the reality of the one have also denied of the other, and conversely. Therefore one will not be surprised, in establishing the reality of space we regard also time as something real, (...) and in this regard I agree that the idea of time does not exist but in our imagination." [Euler E149, § 18 (Uchii)

[89] For the concept of transcreation which had been discussed by Leibniz [Leibniz, Pacidius].

[90] "1. Les principes de la Méchanique sont déjà si solidement établis, qu'on auroit grand tort, si l'on vouloit encore douter de leur vérité. (...) 2. Ces deux vérités étant si indubitablement constantées, il faut absolument qu'elles soient fondées dans la nature des corps; (...)." [Euler E149] "18. Ces axioms se rapportent à des corps infiniment petits, (...) et partant ces formules renferment en soi la première loi du movement, en vertu de laquelle tout corps étant en repos y demeure; or étant en

independence of bodies of all other things, in first respect, the independence of bodies of ghosts [Euler E842, § 49]. The basic distinction between the *res extensa sive corpus* and *res cogitans sive mens* had been introduced by Descartes and the *res extensa* had been mechanically represented by the continuum or plenum.[91]

Following Leibniz, the necessary, i.e. non-contingent and persisting, discretization or distinction of *different* bodies from each another had to be performed by additional principles called monads [Leibniz, Monadolgy, § 8]. Euler claimed that these persisting properties are guaranteed by the internal principles, mainly by the impenetrability [Euler E842, Chaps. 2, 3, 4, 5 and 6]. Moreover, Euler based mechanics of the idea of point like bodies to whom masses are additionally assigned (compare Sect. 4.1.2). Hence, the discretization had been *arithmetically* represented if it is assumed that the bodies consists of parts being such mass points [Euler E015/016, §§ 134–141] and, straightforwardly, if the relations between bodies are expressed in terms of their masses.

> Cum enim nullum temporis punctum sit, quo corporum non existat, quin continuo existat, dubitari nequit, haecque continua corporum existentia in motu aeque atque in quiete concedi debet. [Euler E 289, § 23]

The relation between straight and curvilinear motion obtained analytically can be represented in terms of geometry and mechanics. Every curvilinear motion is due to forces, but not every straight motion is independent of forces. The reason is that a force acting in the direction of motion does not change this direction, although it changes the state of the body. Therefore, the geometrical model is not sufficient to

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mouvement le corps continue uniformément selon la même direction, à moins qu'il ne soit sollicité par quelque force de dehors." [Euler E177] The latter statement refers to the external principles. Euler obtained a *unified analytical representation* of internal and external principles. "1. C'est une propriété générale de tous les corps, que personne ne révoque plus en doute, que chaque corps considéré en lui-même demeure constamment dans le même état, ou de repos ou de mouvement." [Euler E181, § 1] Following Descartes and Newton [Newton, Principia, 1st Law], the notion of "substance" had been translated into the notion "considéré en lui-même".

[91] Bodies cannot violate the laws of mechanics. All changes observed in the realm of bodies are necessary and result from the action of forces [Euler E417, Lettre LXXXV] (compare the discussion of the different laws proposed by Daniel Bernoulli by Euler and the interpretation of the necessary relations between the change of velocity and the forces [Euler E015/016, § 152]). Daniel Bernoulli published the proposed formula in 1726, Euler discussed the derivation in 1736 and d'Alembert commented on Euler's demonstration in 1743. In 1765, Kästner summarized the results of the debate about a problem which is seemingly not satisfactorily solved: "73. Anm. Auf dieser nun gefundnen Gleichung beruht alles, was sich von Wirkungen anderer Kräfte als unserer Schwere sagen lässt. Sie selbst aber beruht auf (65). (...) Die grössten Lehrer der Mechanik finden hier Schwierigkeiten. Herr Daniel Bernoulli hat den Ausdruck (72) unter die blos zufälligen Wahrheiten gerechnet und geglaubt, es könnte auch eben so gut $du = f^* 2\,dt$ u. s. w. seyn (*). Daher hat sich Hr. Euler bemüht einen Beweis zu geben (**), damit aber Hr. D'Alembert nicht überzeugt (***). Der letztere, will die Gleichung (72) als eine Erklärung dessen, was man unter einer beschleunigenden Kraft versteht annehmen. (...) Daß man aber bey de Betrachtungen des Druckes und der Bewegung hier vereinigen müsse, hat Hr. Euler sehr richtig erinnert (****). Wenn mir mein Verfahren gelungen ist, so wird es zugleich Cartesens Grundsatz der Mechanik (&) erläutert haben. (*) Examen principiorum Mechanicae Sect. I. § 1. Comm. Ac. Imp. Petropol. T. I. p. 127. (**) Mechanica L. I. §§ 146–152. (***) Dynamique art. 19. (****) l. c. § 213. (&) Mechanica in R. Des Cartes Opusc. Posthumis (Amstelod. 1701. 4)." [Kästner, Anfangsgründe, § 73]

describe completely the difference between the change of the state and the change of the place [Euler E417, Lettre LXXIV].[92]

4.1.5 *The Relations Between Straight and Curved Lines and Paths*

Euler based the analytical representation of motion on the difference between uniform and non-uniform motion. In case of a non-uniform motion, the smallest elements of path are traversed by a uniform motion [Euler E015/016, § 33]. Generally, Euler distinguished three types of motion belonging to different geometrical objects, (i) the straight line, (ii) the plane and (iii) the space (solidum).

> Cognitio ergo huius spatii tres casus revocatur, quorum primus est, si motus sit rectilineus spatiumve linea recta. Secundus, si spatium quidem sit linea curva, sed tota in eodem plano sita. Tertius vero, si linea curva non eodem plano contineatur. [Euler E289, § 22]

Straight and curved motion can be only distinguished in cases (ii) and (iii). Hence, geometrically, motion is expressed in terms of the well-known model where the "elements of curves are treated as infinitely small straight lines". Every non-uniform motion is decomposed into "infinitely small uniform motions". Euler did not introduce different notions for finite and infinitesimal motions (translations), but transferred the characteristic features observed for bodies of finite size and distinguished shape travelling finite distances Δs in finite time intervals Δt to infinitesimal distances ds and infinitesimal time intervals dt. The transfer is based on the Leibnizian principle of continuity completed by the arithmetical representation of the *temporal* and *spatial intervals*. The basic items are not *instants* and *positions*, but always intervals and the relations between intervals. The representation of mechanical laws of non-uniform motion of bodies by discrete quantities is due to Galileo who related discrete finite temporal intervals to discrete finite spatial intervals and expressed the differences between the intervals in terms of a series of rational number [Galileo, Discorsi, Third Day].

Following Euler, all kinds of motions are traced back to uniform motions and all kinds of bodies are traced back to bodies of infinitesimal magnitude [Euler

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[92] The progress Euler made was only partially acknowledged by his contemporaries. For the purpose of clarification, the work of Kästner is appropriate since he refers explicitly to Euler's *Mechanica*. Kästner's introduction into mechanics is very elucidating. Kästner claimed that gravitation is the prototype of a force. Kästner stressed that he followed the procedure Euler had given in the *Mechanica*. "Da ich aus Hrn. Eulers Buche, als es noch ganz neu war, diesen Theil der höheren Mechanik durch eigenen Fleiß erlernt (...)." [Kästner, Anfangsgründe, Preface] It is very likely that Châtelet also made use of Euler's treatise in writing the *Institutions*. Moreover, Châtelet sent the *Institutions* to Euler and Euler mentioned Châtelet's *Institutions* in the correspondence with Goldbach [Reichenberger]. Euler sent two letters to Châtelet. "En lisant vos Institutions Physiques, j'ai également admiré la clarté, avec laquelle Vous traitez cette science, qui la facilité, avec laquelle Vous expliquez les choses les plus difficiles sur le mouvement, qui sont même assez embarassantes, quand il est permis de se servir du calcul." [Euler, Correspondence with scholars, p. 275].

E015/016, § 33] which are purely progressive, i.e. described by translations and, moreover, by an invariant velocity.

> Ces axioms se rapportent à des corps infiniment petits, ou tels, qui ne soient susceptibles d'autre mouvement que de progressif, et c'est de là que tous les autres principes du mouvement doivent être déduits, (...) dont les corps sont composées des élémens, (...) et partant ces formules renferment en soi la première loi du mouvement, en vertu de laquelle tout corps étant en repos y demeure; or étant en mouvement le corps continue uniformément selon la même direction, à moins qu'il ne soit sollicité par quelque force de dehors. [Euler E177, §§ 18 and 23]

All other kinds of motions are composed of one, two or three elementary progressive motions or translations. These are the basic elements which are modified in the presence of forces or "external causes" which are described in terms of forces generated due to the interaction of bodies [Euler E181], [Euler E289, § 131].

> 131. Si duo corpora ita coeunt, ut neutrum statum suum conservare possit, quin per alterum penetret, tunc in se mutuo agunt viresque exerunt, quibus eorum status mutetur. [Euler E289, § 131]

This theorem may be considered as the central column Euler founded the whole construction of the theory. The interacting bodies form a whole, i.e. a new system, whose properties are essentially different from the systems of non-interacting bodies. The forces are exclusively generated by the interaction of bodies. Action at distance is excluded. Euler developed an alternative model of gravitation by the assumption tat the "heaviness is not due the mutual attraction of bodies, but it is generated by the pressure of ether" [Euler E842, Chap. 19].

Euler modelled the differences by different kinds of "world systems" (compare Sect. 4.4). From the model for the origin of forces presented after 1750, it follows (i) that the bodies are acting mutually and (ii) that the action is not instantaneous, but persists in the time interval the interaction is performed.

In the *Mechanica*, Euler still distinguished between the beginning and the duration of time and assumed that in the case of non-uniform motion "the whole change of velocity takes place at the *beginning* of the single elements" [Euler E015/016, § 34].[93] Then, independently of the length of the time interval, the non-uniform motion is resolved into a succession of discrete instantaneous events where the force is

[93] This model for the change in motion had been discussed by Hobbes and Leibniz within the model of the curved line and the inscribed polygon (compare the comments of Arthur [Arthur, Syncategorematic] and Meli [Meli, p. 83]). "Implicit in Leibniz's reasoning here, of course, is the assumption of a direct correspondence between curves understood as trajectories of physical bodies, and curves as mathematical objects. If a body follows a continuously curved trajectory only in a fictitious limit as it is bombarded by ever finer particles in the plenum, this must correspond to the fact that the mathematical curve arises only as a fictitious limit of polygons of ever more numerous, but always finite, sides. (*see below) This, in turn, would require the provision of a justification for infinitesimals on strictly Archimedean grounds. If infinitesimalist techniques could be justified by means of a proof based on the Archimedean axiom, then infinitesimals could be interpreted as fictions, standing for the fact that such a finitist technique could always be substituted for them. A sketch of just such a justification is given by Leibniz in a piece dated March 26th 1676 and titled by the Akademie editors De infinite parvis: 'We need to see exactly whether it can be demonstrated in quadratures that a difference [differentia] is not infinitely small, but rather nothing at all [omnino nulia]. And this will be shown if it is established that a polygon

acting which are separated by gaps where the force is not acting.[94] Obviously, this model is a heritage of the 17th century discussions of motion (compare Chap. 2) and an incomplete foundation of the calculus. The change in velocity is represented by the expressions

$$c + dc \quad \text{and} \qquad c + 2 \cdot dc + ddc \tag{4.2}$$

with $c + dc + d(c + dc) = c + 2 \cdot dc + ddc$. The terms are interpreted as the increment dc (differential) and the increment of the increment (differentio-differential) ddc of velocity, respectively.

$$c + \dot{c} \cdot do \quad \text{and} \qquad c + 2 \cdot \dot{c} \cdot do + \ddot{c} \cdot o^2 \tag{4.3}$$

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can always be inflected to such a degree that even when the difference is assumed infinitely small, the error will be smaller. Granting this, it follows not only that the error is not infinitely small, but that it is nothing at all —since, of course, none can be assumed.' (A VI, III, 434; LLC, 64–65) As I have argued elsewhere, this argument is very reminiscent of Newton's justification for his Method of First and Last Ratios. (*) For an insightful discussion of the relationship between the polygonal representation of curves and physical trajectories see D. Bertoloni Meli: Equivalence and Priority: Newton versus Leibniz, Oxford, 1993, p. 83. As Meli observes, 'the vertices of the infinitangular polygon cannot be the places where impacts occur; Leibniz's mathematical representations of curvilinear motion are fictitious'." [Arthur, Syncategorematic]

[94] This model of non-uniform motion is very similar to the model Leibniz had developed for all kinds of motion [Leibniz, Hypothesis] (compare Chap. 2) "This is a nod to the new theory he developed in 1670 under the influence of Hobbes, where the continuity of motion is construed in terms of Hobbesian conatus or endeavours, reconstructed as incorporeal 'beginnings' of motion. This theory, elaborated by Leibniz in a series of working papers over the following year, was published in 1671 as the Theoria Motus Abstracti (TMA)." [Arthur, Syncategorematic] As it had been demonstrated by Arthur, Leibniz procedure is intimately connected to the analysis of the continuum and the actual infinity. Arthur claimed that Leibniz interpretation of differentials as fictitious quantities is rooted in these early investigations. Hence, although Euler had no knowledge of Leibniz's paper (it was only published in the 19th century), it can be assumed that he knew Hobbes' theory. Hence, Euler's reinterpretation of differentials in the frame of a mechanical model marked an essential turn in the explication and understanding of their status from "fictitous" to "non-fictitious" quantities. Adopting Leibniz's interpretation as syncategorematic quantities, but declaring the infinitesimals to be "real zeros" with respect to any "finite number", Euler removed the status of "fictitious quantities" since "zero" is by no means "fictitious". "In this paper I attempt to trace the development of Gottfried Leibniz's early thought on the Status of the actually infinitely small in relation to the continuum. I argue that before he arrived at his mature interpretation of infinitesimals as fictions, he had advocated their existence as actually existing entities in the continuum. From among his early attempts on the continuum problem I distinguish four distinct phases in his interpretation of infinitesimals: (i) (1669) the continuum consists of assignable points separated by unassignable gaps; (ii) (1670–1671) the continuum is composed of an infinity of indivisible points, or parts smaller than any assignable, with no gaps between them; (iii) (1672–1675) a continuous line is composed not of points but of infinitely many infinitesimal ones, each of which is divisible and proportional to a generating motion at an instant (conatus); (iv) (1676 onward) infinitesimals are fictitious entities, which may be used as compendia loquendi to abbreviate mathematical reasonings; they are justifiable in terms of finite quantities taken as arbitrarily small, in such a way that the resulting error is smaller than any pre-assigned margin. Thus according to this analysis Leibniz arrived at his interpretation of infinitesimals as fictions already in 1676, and not in the 1700's in response to the controversy between Nieuwentijt and Varignon, as is often believed." [Arthur, Syncategorematic]

Already in *Mechanica* and in all later versions of the theory from 1748 [Euler E842] and 1765 [Euler E289], Euler introduced an essential paradigmatic change in the interpretation of the basic relation between the change in velocity caused by the action of forces. Instead of the previous *instantaneous* action of forces at the *beginning* of a time interval, Euler assumed a permanent action of force during the whole time interval [Euler E015/016, § 130]. The forces are assumed to be constant, i.e. invariant and independent of time and coordinates, and the increment of velocity is not interpreted as generated at the *beginning* of the *element of time*, but during the whole time interval described interms of this *element of time*, called tempusculum. The increment of velocity is proportional to the product of force and element of time,

$$dc \sim K \cdot dt, \tag{4.4}$$

i.e. the increment of velocity dc is related to the whole element of time dt, and the action of force is not restricted to an instant and then interrupted, but the force is present during the whole length of the time interval independently of the difference in magnitude of finite or infinitesimal quantities.[95] On the contrary, Euler assumed that the force preserves the same magnitude and, i.e. in case of a constant force (than there is no doubt that the force is neither temporarily annihilated and recreated), the change in velocity is always proportional to the time the force is acting, independently of the length of the time interval which "may considered as finite or infinitesimal time intervals". This analytical representation follows from the Euler's assumption that the forces are generated by the bodies due to their impenetrability [Euler E181], [Euler E842]. The forces are present as long as the bodies are interacting and the time interval is mechanically determinate as the time of interaction.

[95] "Hier muss die Zeit, so lang der Körper von der Kraft gedrückt wird, nothwendig in Betrachtung gezogen werden; denn wenn ein Druck eine Wirkung hervorbringen soll, so muss derselbe von einiger Dauer sein, so kurz dieselbe auch sein mag. Je länger also ebendieselbe Kraft auf den Körper wirket, je grösser muss die Veränderung sein, welche in dem Zustande desselben hervorgebracht wird; in einer doppelten Zeit wird nämlich die Veränderung zweimal, in einer dreifachen Zeit dreimal so gross sein und so fort. Da wir nun setzen, dass der Körper von der Kraft vorwärts fortgestossen werde, so besteht die Veränderung seines Zustandes in der Vermehrung seiner Geschwindigkeit, und also muss von ebenderselben Kraft die Geschwindigkeit in einer doppelten Zeit einen zweimal so grossen Zuwachs erhalten, in einer dreifachen Zeit einen dreimal so grossen und so fort; das ist, der Zuwachs der Geschwindigkeit, so von ebenderselben Kraft in dem Körper gewirket wird, muss sich wie die Zeit verhalten. Wenn wir demnach die Geschwindigkeit, welche der Körper jetzt hat, durch v andeuten und den Zuwachs derselben durch dv, welcher in der Zeit dt gewirket wird, so verhält sich dv wie dt; nämlich in einer anderen Zeit ndt wird der Zuwachs der Geschwindigkeit sein ndv, und dieses ist wahr, man mag die Zeit dt nebst dem inzwischen gewirkten Zuwachs der Geschwindigkeit dv als unendlich kleine Grössen ansehen oder als endliche, wenn nur die Kraft die ganze Zeit über einerlei Grösse behält. Da nun der Zuwachs der Geschwindigkeit, welcher in einer endlichen Zeit hervorgebracht wird, nicht anders als endlich sein kann, so muss der Zuwachs der Geschwindigkeit dv, so in einem unendlich kleinen Zeitpunkt dt gewirket wird, unendlich klein sein. Eine gleiche Bewandniss hat es mit dem Verluste der Geschwindigkeit, wenn der Körper von der Kraft rückwärts gedrückt wird; alsdann aber wird derselbe Verlust durch $-dv$ ausgedrückt, und verhält sich also $-dv$ wie dt." [Euler E842, § 52] The same supposition is introduced for spatial intervals [Euler E842, § 59].

> 52. Wenn ein bewegter Körper von einer Kraft vorwärts getrieben wird, so ist der Zuwachs der Geschwindigkeit um so viel grösser, je länger diese Kraft auf den Körper wirket, und ebenso verhält es sich mit dem Verlust der Geschwindigkeit, wenn die Kraft rückwärts auf den Körper wirket. [Euler E842, § 52]

The generalization to the 2D and the 3D case is analytically performed by simply increasing the number of independent variables and adding similar equations for the force components *P,Q,R*

$$du \sim P \cdot dt, \quad dv \sim Q \cdot dt \qquad dw \sim R \cdot dt \tag{4.5}$$

where the changes of the components of velocity *du,dv,dw* are related to the spatially determined differentio-differentials ddx, ddy, ddz, respectively, by the equations $du = \frac{ddx}{dt}, dv = \frac{ddy}{dt}, dw = \frac{ddz}{dt}$. The advantage of the Eulerian representation is that all expression had to be geometrically interpreted as straight motions and the question to explain the change in the direction in the corners of the polygon does not appear (Hobbes-Leibniz[96]) (compare Chap. 2). Following Galileo (compare Chap. 1) and representing the *independent variable time* by an arithmetic progression $t, t+\Delta t, t+2\Delta t, t+3\Delta t, t+4\Delta t,\ t+5\Delta t\ldots$ of increments, i.e. multiples of the quantity Δt, there are neither finite nor infinitesimal time intervals between two consecutive parts of the arithmetic progression. The continuity of time is represented in terms of series of positive and negative integers and consecutive discrete time intervals. The principle of continuity had been generalized by Euler [Euler E212, §§ 95–102] (compare Chap. 5). Euler demonstrated that the general relations between mechanical quantities are preserved if the finite increments are replaced with infinitesimal increments in terms of multiples of dt. Following Euler, the *differential* calculus is a special case of the *calculus of differences* [Euler 1727], [Euler E212, §§ 112–114] (compare Chap. 5). Euler composed mechanics by a synthesis of *discrete* and *continuous* quantities. The geometric ratio between the discrete increment of a function Δf and the discrete increment of the independent variable $\Delta\xi$ is expressed in terms of continuous functions $g(\xi), h(\xi), k(\xi), \ldots$, i.e. by the relation

$$\frac{\Delta f}{\Delta\xi} = g(\xi) + h(\xi)\Delta\xi + k(\xi)\Delta\xi^2 + \ldots. \tag{4.6}$$

The analytical representation of mechanical relations follows from the mechanical interpretation of the function $f(\xi)$, the independent variable ξ and the increment of the independent variable $\Delta\xi$. Following Euler, the variable and the increment are scalars which can be represented by numbers obtained from experiment, e.g. by measuring a time difference or a spatial distance. Analytically, the independent variable is given by the arithmetical progression $t, t+\Delta t, t+2\Delta t, t+3\Delta t, \ldots$ whereas the function $f = f(t)$ can be chosen arbitrarily describing either the time dependent *path* or the time dependent *velocity* or any other mechanical quantity. The analytical structure of the expression

[96] Compare Meli's comment [Meli, p. 83].

$$\frac{\Delta f}{\Delta t} = g(t) + h(t)\Delta t + k(t)\Delta t^2 + \dots \tag{4.7}$$

is preserved, but the *mechanical interpretation* of the functions $g(t), h(t), k(t), \dots$ becomes different. Choosing the path as to be described by $f = f(t) = x(t)$, the function $g(t) = v(t)$ becomes necessarily a velocity whereas choosing the velocity as to be described by $f = f(t) = v(t)$, the function $g(t) = b(t)$ becomes necessarily an acceleration which may be represented by the geometric ratio of force and mass K/m. Following Euler, an one to one correspondence between the functions f and g whose validity is independent of their different mechanical interpretations, but is solely guaranteed by the analytical representation, is obtained if one got rid of all other functions $h(t), k(t), \dots$ appearing in Eq. (4.6) except $g(t)$. Then, the analytical representation is a *necessary* condition for the *compatibility* of the different mechanical interpretations or, the mechanical quantities are only compatible with each other if an appropriated analytical representation can be constructed.[97] Thus, d'Alembert's objection can be refuted. Following Euler (compare Chap. 5), the representation of mechanical laws by differentials is justified by the circumstance that the "differential calculus is a special case of the calculus of differences" [Euler E212, §§ 112–114]. This special case is constructed by the specification of the finite increment Δt to become an infinitesimal dt which is represented by a differential. Moreover, any special case of a theory cannot not be valid provided that the theory is valid in the general case. The resulting expression

$$\frac{df}{dt} = g(t) + h(t)dt + k(t)dt^2 + \dots \tag{4.8}$$

is readily reduced and an *exact* relation

$$\frac{df}{dt} = g(t) \tag{4.9}$$

is obtained (compare Chap. 5). Following Euler, Eq. (4.9) is not only *approximately* valid, but is an *exact* or *rigorous* result. Using the terminology of the 18th century, the results is *necessarily* valid [Euler E015/016, §§146–152][98] since the higher order terms $h(t)dt + k(t)dt^2 + \dots$ do not contribute to the geometric ratio df/dt since they are infinitely small in comparison to the term $g(t)$. Following Euler, they are equal to zero, i.e. their contribution is not only neglected due tom their smallness, but vanishing [Euler 1727], [Euler E212, Chaps. I, II, III and IV].[99] Following d'Alembert,

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[97] Hence, d'Alembert's criticism of Euler's derivation of the basic relations fails to touch the crucial point. Separating the relation between the change of velocity and forces from the calculus of differences, d'Alembert claimed that the relation between acceleration and forces is not a necessary relation or an axiom, as Euler claimed, but only a definition (compare also [Kästner, Anfangsgründe, § 73]).

[98] *Necessary* is opposed to *contingent*. [Leibniz, Monadology, §§ 29–37] The prototype for necessary relations is found in geometry.

[99] Leibniz preferred an interpretation that the relation between differentials and other quantities can be modelled by the relations like those of a grain of dust, the diameter of the earth and the distance

the relation (4.9) may be only considered as an appropriate definition of the "accelerating force". D'Alembert claimed that "the word 'accelerating force' denotes that quantity to which the increment of velocity is proportional" [d'Alembert, Traité, § 22].[100] Hence, the controversy on the foundation of mechanics can be traced back to the question whether or not the calculus is a method commensurate and equivalent in rigour to the rigour known for geometric demonstrations (compare Chap. 5). Not suprisingly, d'Alembert also developed a foundation of calculus based on the idea of limits [d'Alembert, Encyclopédie].[101]

From the very beginning, Euler know that an affirmatively answer of this question is only possible if the corresponding proofs are completely independent of geometry and are to be exclusively formulated in terms of algebra and arithmetics. Hence, Euler abandoned any figures [Euler E212, Preface]. Later, Lagrange observed the same procedure in developing mathematics and mechanics in parallel to attain an analytical foundation of mechanics [Lagrange, Mécanique], [Lagrange, Works 11], [Lagrange, Works 12]. Although based on different approaches in the foundation of the calculus, Euler, d'Alembert and Lagrange invented the reliable basis for all further developments of physics in the 19th and 20th centuries. Euler gave an arithmetical foundation with the inclusion of infinitesimal and infinite quantities, d'Alembert

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to fixed stars. "And when we compare an ordinary term, an infinite term, and one infinitely infinite, it is exactly as if we compare, in increasing order, the diameter of a grain of dust, the diameter of the earth, and that of the sphere of the fixed stars." [Leibniz, 1712, Acta Erud. GM V, 389]

[100] "(...) nous nous contenterons de le prendre pour une définition, & d'entendre seulement par le mot de force accélératrice, la quantité à laquelle l'accroissement de la vitesse est proportionnel." [d'Alembert, Traité, § 22] Obviously, this definition is equivalent to the relation between "the change in motion" or the increment of velocity and "the impressed moving force" [Newton, Principia, Axioms] (compare Chap. 2).

[101] "One magnitude is said to be the *limit* of another magnitude when the second may approach the first within any given magnitude however small, although the first magnitude may never exceed the magnitude it approaches." [d'Alembert, Encyclopédie, Limit] In his approach introducing the idea of limit, d'Alembert followed Newton and generalized Newton's theory of "first and ultimate ratios" whereas Euler generalized Leibniz's foundation. "What concerns us most here is the metaphysics of the *differential* calculus. This metaphysics, of which so much has been written, is even more important and perhaps more difficult to explain than the rules of this calculus themselves: various mathematicians, among them Rolle, who were unable to accept the assumption concerning infinitely small quantities, have rejected it entirely, and have held that the principle was false and capable of leading to error. (...) Leibniz, was embarrassed by the objections he felt to exist against infinitely small quantities, as they appear in the *differential* calculus; thus he preferred to reduce infinitely small to merely incomparable quantities. This, however, would ruin the geometric exactness of the calculations; is it possible, said Fontenelle, that the authority of the inventor would outweigh the invention itself? (...) Newton started out from another principle; and one can say that the metaphysics of this great mathematician on the calculus of fluxions is very exact and illuminating, even though he allowed us only an imperfect glimpse of his thoughts. He never considered the *differential* calculus as the study of infinitely small quantities, but as the method of first and ultimate ratios, that is to say, the method of finding the limits of ratios. Thus this famous author has never differentiated quantities but only equations; in fact, every equation involves a relation between two variables and the differentiation of equations consists merely in finding the limit of the ratio of the finite differences of the two quantities contained in the equation. Let us illustrate this by an example which will yield the clearest idea as well as the most exact description of the method of the *differential* calculus." [d'Alembert, Encyclopédie, Differential]

introduced also an arithmetical foundation by the idea of limit to exclude both these quantities whereas Lagrange rejected both the infinitesimal quantities and the concept of limit [Lagrange, Fonctions].[102] All the foundations are guided in goal and spirit by the confidence in the arithmetical operations [Klein, Arithmetization].

4.1.6 The Analytical Representation of Motion

The analytical representation of motion by Euler is rich of details which may presently regarded as trivial since they are commonly known for the contemporary reader. Nevertheless, Euler had to confirm that the principles of mechanics are compatible with the foundation of the calculus he had chosen some years ago in 1727 [Euler 1727]. In 1736, Euler presented an almost completely new subject analyzed by a completely new method which had been never treated before him. Euler is seeking for rigorous demonstration to make the propositions clear and the conclusions sure. The theorems should be "not only true, but necessarily true", i.e. all alternatives are excluded since they involve contradictions.

> 152. Apparet igitur non solum verum esse hoc theorema, sed etiam necessario verum, ita ut contradictionem involveret ponere $dc = p^2 dt$ vel $p^3 dt$ aliumve functionem loco p. Quae omnes cum Clar. Dan. Bernoullio in *Comment*. Tom. I. aeque probabiles videantur, de rigidis harum propositionum demonstrationibus maxime eram sollicitus. [Euler E015/016, § 152]

Here, Euler made use of the rules of logic, the exclusion of contradictions, to reject other relations between the change of velocity and forces. All that what includes a contradiction or results in a contradiction is not possible. By this methodology, Euler rejected the Leibniz-Wolffioan construction of monads as least parts of a body [Euler E081].

Euler stated that the power of the previous demonstrations given in §§ 34 and 35 of the *Mechanica* are based on the supposition that the "change of velocity which appeared while the body is travelling an infinitely small element, is infinitely small and evanesces with respect to the velocity of the body" [Euler E015/016, § 36].[103]

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[102] "On a transporté dans l'analyse les principes qui résultaient de ces considération, (...) envisagés analytiquement, se réduisent simplement à la recherche des fonctions dérivées qui forment les premiers termes du développement des fonctions données, ou à la recherche inverse des fonctions primitives les fonctions dérivées. (...) Maclaurin et d'Alembert emploient la considération des limites et regardent le rapport des différentielles comme la limite du rapport des différences finies, lorsque ces différences deviennent nulles. Cette manière de représenter les quantités différentielles ne fait que reculer la difficulté; car, en dernière analyse, le rapport des différences évanouissantes se réduit encore à celui de zéro à zéro." [Lagrange, Fonctions] Hence, Lagrange rejected Euler's and d'Alembert's approaches by the same reason, the appearance of the indeterminate ratio "zero divided by zero".

[103] Arthur discussed the development of Leibniz's interpretation of the infinitesimal quantities between 1671 and 1714 and accentuated the relation to the continuum problem. "In this paper l attempt to throw light on these issues by exploring the evolution of Leibniz's early thought on the status of the infinitely small in relation to the continuum." [Arthur, Syncategorematic]

Furthermore, it is impossible that a finite velocity is created in an instant since the creation had to have to be repeated in each instant. The technical term "evanescent" used by Euler is related to Leibniz's idea that the differentials are "evanescent quantities" [Leibniz, Specimen, II (4)].[104] Rest is an evanescent motion.

> 36. Demonstrationis datae vis hoc nititur fundamento, quod celeritatis mutatio, quae fieri potest, dum elementum infinite parvum percurritur, debeat esse infinite exigua et evanescere prae celeritate, quam corpus iam habet; (...). [Euler E015/016, § 36]

The change of velocity is smaller than any finite velocity. Hence, the change is independent of the actual value of velocity *assigned* to the body[105] as small or as great it may be in magnitude.[106] In case of a non-uniform motion along a curved line, Euler constructed (i) the path if the velocity and the time are given, i.e. if the time is considered as independent variable, and (ii) the time belonging to motion if the path and the velocity are given [Euler E015/016, § 37].

The elements of the path are traversed uniformly with the velocity c. Then, the time needed for travelling is given by ds/c and, consequently, the integral $\int ds/c$ represents the whole time the body needs to travel through the whole space. Obviously, the expression $ds/c \neq 0$ is assumed to be different from zero.[107] The relation between *infinitesimal* time elements dt and *infinitesimal* path (space) elements ds are arithmetically represented by the same relations (algorithm) being valid for finite temporal and spatial intervals. In case of uniform motion along a straight line, there is no difference in velocity, i.e. the velocity c_{unif} in the representations $\Delta s = c_{\text{unif}} \cdot \Delta t$ and $ds = c_{\text{unif}} \cdot dt$ is of the same magnitude with $c_{\text{unif}} = \text{const}$. In case of *non-uniform* motion, the velocity becomes a function of time $c_{\text{non-unif}}(t)$ and $ds = c_{\text{unif}} \cdot dt$ is replaced with $ds = c_{\text{non-unif}}(t) \cdot dt$. Hence, the relation $\Delta s = c_{\text{unif}} \cdot \Delta t$ cannot remain to be valid for a time interval Δt of arbitrary magnitude. The advantage of Euler's foundation of the calculus (compare Chap. 5) is that the time dependence[108] is doubly represented, at first by a continuous variable "time", $-\infty < t < +\infty$, and, at second in terms of temporal intervals being either of finite or infinitesimal magnitude denoted by Δt and dt, respectively, whose magnitude is specified by their mutual relation. The finite interval is decomposed into smaller intervals of *equal*, but arbitrarily in magnitude length by division, $\Delta s = \Delta S/n = (S_2 - S_1)/n$, and, vice versa, any finite interval can be composed of smaller intervals of *equal* length by summation of these intervals, i.e. $\Delta S = \sum \Delta s = \sum_{k=1}^{n} \Delta s_k = n \cdot \Delta s$. This procedure is only *arithmetically*

There are different problems related to the continuum, (i) the continuous flow of time [Newton, Method of Fluxions], [Newton, Quadrature], (ii) the continuity of extension in case of the bodies, (iii) the continuity of motion and (iv) the continuous action of forces upon the bodies.

[104] Following Leibniz, rest is an evanescent motion. However, this motion is not nothing, i.e. mathematically "zero", but a "minimal motion". It is impossible to assign a finite velocity to the body, but it also not appropriate to assign the "velocity" $v = 0$, i.e. "nothing".

[105] This velocity may experimentally be determinate as the ratio of finite spatial and temporal intervals.

[106] Euler commented that the relations between space, time and the scale of velocities is valid for relative an absolute motion [Euler E015/016, § 55]

[107] Compare [Euler E289, §§ 38–47]

[108] Or any other dependence of a function of any other independent variable.

defined and independent of the length of the intervals and their mechanical interpretation. Arithmetically, time and space are treated on an equal footing [Euler E149, §§ 18–20].

The equality of time intervals for an inversion of the direction of motion will be demonstrated if the "body had at the same position the same velocity [Euler E015/016, § 42]. Three problems are discussed: (i) calculate the *velocity* if the scales of time and space are given, (ii) calculate the *time scale* if the scales of space and the velocity are given and (iii) calculate the space scale if the scales of time and the velocity are given [Euler E015/016, §§ 48–55]. The relations are valid for relative and absolute motions [Euler E015/016, §§ 55–70, 77–97], compared to the method of Archimedes [Euler E015/016, § 56], compared to the method of Newton [Euler E015/016, § 71], to the relation between force of inertia discussed by Kepler [Euler E015/016, §§ 72–76] and, finally, to the program for mechanics [Euler E015/016, § 98].

The non-uniform motion is not only related to forces or as caused by forces, but treated analytically as the *counterpart* of *uniform* motion to demonstrate the common mechanical and mathematical principles being valid in both cases. Hence, the basis for the representation of *non-uniform* motion is the previously analyzed *uniform* motion since the common elements of a complete knowledge of any kind of motion are (i) the path the mass point is travelling along and (ii) the determination of position of the point ("Stelle") for each temporal instant ("Zeitpunkt") [Euler E015/016, § 21]. From the definition of a uniform motion in a straight direction, it follows what a non-uniform motion should be [Euler E015/016, § 22]. The velocity can be defined as the geometric ratio of the path and the time the distance between two positions is travelled by the body, $v = \Delta s/\Delta t$ [Euler E015/016, § 23].[109] In case of a curved motion and non-uniform motion along a straight line, it is possible "to imagine a uniform motion which completely coincides ("völlig übereinkommt") with the non-uniform motion at that instant". Hence, "the direction and the velocity of this uniform motion will be *also assigned* to the non-uniform motion", but only for this instant ("für diesen Augenblick").

> 21. Zu einer vollständigen Erkenntniss aber der Bewegung eines Punktes wird nicht nur erfordert, dass man den von demselben beschriebenen Weg anzuzeigen wisse, sondern man muss in diesem Wege für einen jeglichen Zeitpunkt die Stelle bestimmen können, wo sich der bewegte Punkt damals befunden.
> 22. Eine gradlinichte und gleichförmige Bewegung ist, wenn der Punkt sich erstlich nach einer graden Linie beweget und hernach in gleichen Zeiten gleiche Theile dieser Linie durchläuft; woraus zugleich verstanden wird, was eine krummlinichte und ungleichförmige Beilegung sei.
> 23. Bei einer gradlinichten und gleichförmigen Bewegung wird die grade Linie die Richtung der Bewegung genannt; die Geschwindigkeit aber ist das Verhältniss des Weges zu der Zeit, in welcher derselbe durchlaufen wird.
> 24. Ist die Bewegung aber krummlinicht und ungleichförmig, so kann man sich für einen jeglichen Zeitpunkt eine gradlinichte und gleichförmige Bewegung vorstellen, welche in diesem Augenblicke mit derselben völlig übereinkommt und sowohl die Richtung als die

[109] The direction of motion is to be additionally defined for a uniform motion. The velocity is the numerical value of the ratio of path and time.

> Geschwindigkeit dieser letzten Bewegung wird auch für diesen Augenblick der ersteren Bewegung zugeschrieben. [Euler E842, §§ 21–24]

Euler claimed that the finite velocity of a uniformly moving body can be doubly represented, either by the ratio $v = \Delta s/\Delta t$ or by the ratio $v = ds/dt$ of either finite or infinitesimal spatial and temporal intervals,[110] respectively. The idea of uniform motion for *finite* spatial and temporal intervals is assumed to be valid also in case of *infinitesimal* spatial and temporal intervals, "as if the motion with respect to a path of infinitesimal length would be also uniform" [Euler E842, § 24].

Following Euler, the *assignment* of finite quantities to a pair of correlated infinitesimal quantities is mediated by a finite quantity, called "velocity". The geometrical model for the relation between straight and curved lines is not in contradiction, but in conformity with the given interpretation, but suffers losses from the simple, but decisive fact that time, neither as independent variable nor as function, can be defined geometrically as "straight" or "curved", i.e. by reference to the distinction between straight and curved lines. For such distinctions, more than one dimension (or, arithmetically, more than one independent variable) are to be taken into account as it had been demonstrated in the attractive and extensively used model of the polygon and circle positioned in a plane. Hence, geometrically speaking, time is a 1D quantity (or is arithmetically represented by *one independent* variable), Lagrange claimed that in comparison to geometry "mechanics is like a four dimensional geometry" [Lagrange, Fonctions].[111]

For Cartesian coordinates or Euclidean planes, the infinitesimal translation does not depend on the position of the mass point and is defined by

$$ds^2 = dx^2 + dy^2 \tag{4.10}$$

[110] "Man pflegt zu sagen, eine jegliche Bewegung könne für einen einzigen Augenblick als gradlinicht und gleichförmig angesehen werden, eben wie in der Geometrie die unendlich kleinen Theilchen einer jeglichen krummen Linie mit Recht für grad gehalten werden. Weil aber eine unrichtige Erklärung des unendlich Kleinen leicht Schwierigkeiten machen möchte, so habe ich die Sache auf eine andere Art vorgestellt, welches aber auf eines hinauslauft. Hieraus begreift man nun leicht, dass, wenn sich ein Punkt in einer krummen Linie bewegt, die berührenden graden Linien derselben an einem jeden Ort die Richtung der Bewegung anzeigen; hernach wird durch die Differential-Rechnung die Geschwindigkeit gefunden, wenn man das Differentiale des Weges durch das Differentiale der Zeit theilet; eben als wenn die Bewegung durch einen unendlich kleinen Weg gleichförmig wäre. Es kann also sein, dass bei einer Bewegung sowohl die Richtung als die Geschwindigkeit alle Augenblick verändert werde; man sieht aber deutlich, dass die ganze Erkenntniss einer krummlinichten und ungleichförmigen Bewegung darauf beruhe, dass man alle Augenblicke die Richtung und Geschwindigkeit, so dem bewegten Punkte zukommt, anzeigen könne, und hierauf ist auch die ganze Lehre von der Bewegung der Körper gerichtet." [Euler E842, § 24]

[111] "Ainsi, on peut regarder la Mécanique comme une Géométrie à quatre dimensions et l'Analyse mécanique comme une extension de l'Analyse géométrique." [Lagrange, Fonctions, Part III, § 1] In the 20th century, Minkowski demonstrated that the independence of space and time can be only removed by establishing an upper limit for velocity. "The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." [Minkowski, Space and time, p. 75] The *upper limit of velocity* cannot be derived from the properties of space and time, but the relations between space and time follow from the existence of an upper limit of velocity.

[Euler E842, Chap. 8] being the basic formula to describe the preservation of direction and change in direction by $dx = dy$ and $dx \neq dy$, respectively. Analytically, the time variable has now the same status as the spatial variable (coordinates) either as an independent or as a dependent variable. The time variable can be augmented by temporal increments $t, t+\Delta t, t+2\Delta t, \ldots$ independently of the augmentation of coordinates by spatial increments $x, x+\Delta x, x+2\Delta x, \ldots$. The time is resolved into a series of time intervals of equal length, but being arbitrary in magnitude. It is always possible to correlate a spatial interval of arbitrary length with a temporal interval of as arbitrary length since there is no *upper limit* of velocity in mechanics.

As it will be demonstrated, this universal relation between spatial and temporal intervals of arbitrary length can be traced back to Euler's assumption that the "increment of velocity is independent of velocity" [Euler E015/016, § 131]. The increment of velocity is not caused by an increase of the spatial interval or a decrease of the temporal interval, but due to the forces generated in the interaction of bodies. Hence in the next chapter, Euler discussed the motion of bodies in the presences of forces.

4.1.7 External Principles: Forces

The basic concepts Euler had been introduced until now are (i) rest and motion including absolute and relative motion,[112] (ii) the priority of relative motion [Euler E015/016, §§ 7 and 97], (iii) velocity, (iv) uniform and (v) non-uniform motion. The relations between (iv) and (v) had been discussed in the §§ 1–97. Now, Euler discussed the relations between rest and motion, i.e. the change of rest into motion and the change of uniform into non-uniform motion.

> Potentia est vis corpus vel ex quiete in motum perducens vel motum eius alterans. [Euler E015/016, § 99]

In both cases, the cause of the change is a force. The force is the puissance (external power, potentia) which either is leading (perducens) the body from rest to motion or alters its motion. An example for such force is the gravitation (compare Kästner on the gravitation as prototype for forces [Kästner, Anfangsgründe]). The *preservation of state* is presented as a Corollary in § 100 [Euler E015/016, § 100]. The force is the cause for the change of the state of the bodies. The force is called that whatever disturbes the state of the bodies ("quicquid enim corpus de statu suo deturbare valet, potentiam appelamus", literally, prevails to drive off the body (out of his status)), i.e. the force is getting predominance over the body. The action of the force

[112] Following Descartes and regarding only relative and relative motion, rest and motion are *different states* of the bodies and are not in *logical opposition* to each other, i.e. it is not necessary to introduce different mechanical concepts for rest and motion. Both the states are properly described by the same mechanical notion, the inertia of bodies [Euler E015/016], [Euler E842]. The only excluded case is that a body A is resting and moving with respect to the same body B whereas A can be at rest to C while it is moving relatively to D.

is not only a "disturbance", but even more, it is something like as an enforcement coming from outside, i.e. from other bodies [Euler E015/016, §§ 99 and 100]. In the next paragraph, Euler discussed the difference between statics and mechanics Euler [Euler E015/016, § 101] and added, following Newton, that he did not discuss the origin of forces, i.e. whether such forces have their origin in the bodies or are given per se in the world ("an vero per se tales denture in mundo") [Euler E015/016, § 102].[113]

The novelty Euler introduced in comparison to Newton is, firstly, following Newton, the investigation of the change in motion (explicitly presented in the *Anleitung* [Euler E842, §§ 51–56]) and, but secondly investigating independently of motion the change of rest (explicitly presented in the Anleitung [Euler E842, §§ 57–62]). In the 2nd Law, Newton determinate the condition for "change of motion" ("mutationem motus proportionalem esse vi motrici impressae" [Newton, Principia, Axioms]). Euler assumed that the action of a force like the gravitation generates increments dv of the same magnitude for resting and moving bodies [Euler E015/016, § 111].

> 111. Potentia absoluta est potentia, quae in corpus sive motum sive quiescens aequaliter agit. [Euler E015/016, § 111][114]

In Newton's 1st Law, the states of rest and motion are treated on an equal footing, but in the 2nd Law this equivalence is not preserved sinc the 2nd Law is formulated with respect to the change in motion. Euler preserved the equivalence between rest and motion as states of the bodies by the equivalent treatment of the changes of these states (by external causes). This approach is systematically connected to the basic parts of Euler's program to demonstrate the science of motion analytically[115] and already manifested in the formulation of the Proposition 14 where Euler supposed that the "influence of a force on a resting body is known". Then, the effect of the same force on a moving body is to be investigated, i.e. the rules for the change of rest are *transferred* to the rules being investigated for the change of motion. This procedure is advantageously applied to the "very beginning of motion" where any changes or increments are necessarily of infinitesimal magnitude before becoming finite [Leibniz, Specimen, I (6)][116] (compare Chap. 2) which had been investigated by Newton.

> 118. Dato effectu potentiae absolutae in punctum quiescens, invenire effectum eiusdem potentiae idem quomodocunque motum. [Euler E015/016, § 118]

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[113] Contrary to Newton, Euler finally decided to give an answer in a later paper entitled *Recherches sur l'origine des forces* [Euler E177]. Hence, it may be justified to guess that he had at least a preliminary answer already in 1736.

[114] A relative force is observed for a swimming body in a river. This force only appeared for different velocities of the water and the body.

[115] Applying the geometrical method, Newton compared the path, i.e. the distance represented by a straight line, a body is travelling by uniform motion to the "errors" appearing due to the action of forces represented by the deviation from that straight line [Newton, Principia]. In case of rest, there is no frame of reference to determine geometrically the errors.

[116] Leibniz modelled this stage of motion by the relation between dead and living forces.

The forces are not given in advance as in case of "impressed moving forces", but are generated by the interacting bodies to prevent the mutual penetration.[117] The increments of velocities are given in terms of body-force relations which had been introduced by Newton and described by "moving impressed forces". In § 102, Euler stated that he did not intend to analyze the problem whether the forces are present in nature or are related to bodies.

> 102. Utrum huiusmodi potentiae ex ipsis corporibus origenem suam habeant, an vero per se tales dentur in mundo, hic non definio. [Euler E015/016, § 102]

The prototype of forces is the gravitation whose influence demonstrates the reality of forces.[118] In the same manner, the paths of planets should be not curved, but straight if there would not be forces. The relations analyzed in Sect. 4.1.5 describe a correlation between the change in velocity and the presence of forces which had been already postulated by Newton: "The change in velocity is proportional to the impressed moving force" [Newton, Principia]. As it had been demonstrated the relations are incomplete since the origin of forces remains to be indeterminate as it had been stated in § 102. Following Leibniz and assuming that the forces are manly due to the impact of bodies, the relations discussed in Sect. 4.1.5 are to be completed by the inclusion of at least *one additional* body. Then, it follows that for each of the bodies denoted by B1 and B2 the changes of velocity are described by the increments dv_1 and dv_2. The increments

$$dv_1 \sim K_{12} \cdot dt, \qquad dv_2 \sim K_{21} \cdot dt \tag{4.11}$$

$$A \cdot dv = -P \cdot dt(\mathrm{a}), \quad B \cdot du = P \cdot dt(\mathrm{b}) \tag{4.12}$$

depend on forces being equal in magnitude, but opposite in direction. Hence, a difference between the increments dv_1 and dv_2 cannot be explained by the forces.[119] The relations are completed by the introduction of the masses A and B. Euler assumed that the increment of velocity is proportional to the inverse ratio of masses, i.e. $dv_1 \sim 1/A$ and $dv_2 \sim 1/B$. This relation is independent of the magnitude of forces (external principles), but is only due to the interaction independent properties of bodies (internal principles). The forces are generated by the bodies due the interaction. Making use of the Eq. (4.12 (a) and (b)) Euler obtained the important relations[120]

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[117] "(...) wir werden aber weiter unten sehen, dass dies keineswegs mit Recht geschehen kann; denn obgleich alle Körper ohne Zweifel ausgedehnt sind, so folget nicht, dass alle Dinge, welche ausgedehnt sind, sogleich zu Körpern werden." [Euler E842, § 9] "35. (...) und ungeachtet, dass Cartesius das Wesen der Körper in der blossen Ausdehnung gesetzt, so hat er geglaubt, dass die Undurchdringlichkeit mit der Ausdehnung verbunden sei. (...) 49. *Alle Veränderungen, welche in der Welt an den Körpern vorgehen, insofern dazu von Geistern nichts beigetragen wird, werden von den Kräften der Undurchdringlichkeit der Körper hervorgebracht, und finden also in den Körpern keine anderen als diese Kräfte statt.*" [Euler E842, §§ 35 and 49]

[118] Also Kästner presupposed the gravitation as a prototype for the modelling of other forces [Kästner, Anfangsgründe].

[119] Here, Euler generalized Archimedes' model for the equlibrium of the lever.

[120] "40. Ici il est d'abord clair qu'en ajoutant ces deux équations on aura (...) équation qui est indépendante de la force P et qui auroit également lieu, quand même cette force n'auroit pas sa

$$A \cdot dv + B \cdot du = 0(\text{a}), \qquad A \cdot v + B \cdot u = \text{const (b)} \tag{4.13}$$

which are independent of the forces. They follow from the equality of the magnitudes of the terms $|P \cdot dt|$ in Eq. (4.12) and the subsequent integration [Euler E181, § 40], respectively. The relation is interpreted as the conservation of the centre of gravity.[121] The existence of the constant follows from the mathematical requirements of differentiation and integration, generally from the rules of the calculus. The *mechanical* interpretation is due to the definition of masses (inert masses) and forces. All notions are related to a "world" or a model system which is made up of the two bodies and the empty space. Obviously, it is not mandatory to introduce quantities like du/dt or dv/dt. The mechanical interpretation follows from the distinction between the differentials du and dv and the corresponding integrals $\int du$ and $\int dv$. The mechanical properties are not modified by the mathematical operations because all the quantities $[du] = [\text{velocity}]$, $[dv] = [\text{velocity}]$, $[\int du] = [\text{velocity}]$ and $[\int dv] = [\text{velocity}]$ are of the same dimension of a *velocity* because the increment of a velocity and the sum of velocities are necessarily also of the dimension of velocity.

Hence, the *mechanical homogeneity* and, consequently, the validity of the mechanical relations, is guaranteed provided that the *mathematical homogeneity* had been also assured.[122] The model can be readily transferred from the representation by finite quantities into the representation by infinitesimal quantities. As it will be demonstrated in Chap. 5, there are common terms of *finite* magnitude in both representations[123] whose mechanical meaning is the same in both representations.

Having derived the general law of the conservation of the centre of gravity [Euler E181, § 40], Euler discussed the problem to determine the unknown velocities u and v. There is only one equation for two unknown quantities. Therefore, an additional

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grandeur déterminée, que l'évitation de la pénétration exige. Donc à chaque instant que dure le choc, la valeur de cette expression A v + B u sera toujours la même et partant aussi égale à cette, qui lui convient au commencement du choc. Or la valeur de cette expression étant alors = A a + B b, il sera à tous les instants que dure le choc Av + B u = A a + B b et cette équation aura aussi lieu à la fin du choc, de sorte que si v et u marquent les vitesses des corps après le choc, il soit aussi A v + B u = A a + B b, ce qui est une propriété généralement reconnue dans tous les chocs de corps, et qui est comprise dans ce grand principe, que le mouvement du centre commun de gravité n'est altéré par action, que les corps soutiennent dans le choc." [Euler E181, § 40]

[121] Later, Lagrange demonstrated that the basic principles of mechanics are obtained from a general formula by the application of the variational principle. These mechanical principles are the following: (i) the conservation of living forces, (ii) the conservation of the center of gravity, (iii) the conservation of angular momentum and (iv) the principle of least action. "13. Un des avantages de la formule dont il s'agit est d'offrir immédiatement les équations générales qui renferment les principes ou théorèmes connus les noms de *conservation des forces vives*, de *conservation du mouvement du centre de gravité*, de *conservation des moments de rotation* ou *Principe des aires*, et de *Principe de la moindre quantité d'action*." [Lagrange, Mécanique, Seconde Partie, Dynamique, § 13]

[122] For the mathematical homogeneity of the expression in the calculus compare Newton [Newton, Quadrature (Kowalewski), § 1], Leibniz [Leibniz, Homogeneity] and Euler [Euler E212]. Leibniz's paper in entitled "Symbolismus memorabilis calculi algebraici et infinitesimalis in comparatione potentiarum et differentiarum, et de lege homogeneorum transcendentali" [Leibniz, Homogeneity].

[123] In the frame of the calculus, these terms had later been called *standard parts* [Keisler] (compare Chap. 5).

relation between u and v is needed which is obtained from the calculation of living forces nowadays called kinetic energy [Euler E181, §§ 41 and 42]. Using the *same* force P as before,[124] Euler obtained an additional relation between the unknown velocities

$$A \cdot v \cdot dv + B \cdot u \cdot du = -Pdx + Pdy = -Pdz \tag{4.14}$$

whose mechanical and mathematical homogeneity is also guaranteed, but whose mechanical interpretation is different compared to Eq. (4.13). Now, $dx = vdt$ and $dy = udt$ are the spaces the bodies A and B are travelling, respectively, and the integration

$$A \cdot v \cdot v + B \cdot u \cdot u = \text{const} - 2\int Pdz \tag{4.15}$$

results also in a constant, but additionally in a function depending on the path the body is travelling [Euler E181, § 43]. Denoting the velocities of the bodies *before* the interaction by a and b, it follows

$$A \cdot v \cdot v + B \cdot u \cdot u = A \cdot a \cdot a + B \cdot b \cdot b - 2\int Pdz \tag{4.16}$$

and the velocities *after* the interaction can be expressed in terms of the velocities *before* the interaction assuming $\int Pdz = 0$.

$$v = \frac{2Bb + (A-B)a}{A+B} \qquad u = \frac{2Aa - (A-B)b}{A+B} \tag{4.17}$$

The only parameters are the velocities before the impact and the masses of the bodies [Euler E181, § 45] and, the only, but general assumption is that the impenetrability is a common property of all bodies. The impenetrability is of the same magnitude for all bodies[125] independently of the extension and the mass of the bodies, whereas the mass of the bodies is an internal property whose magnitude is different for different bodies.

The 3rd Law of Newton's axiomatic systems[126] becomes a theorem within the frame established by Euler for the relation between internal and external principles.[127]

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[124] In 1686 and later in 1695, Leibniz distinguished between "dead" and "living forces" (compare Chap. 2) and initiated the debate on the measure of living forces. The debate was finally terminated by the demonstration that it is the *same force* which is either related to the time it is in action or to the space the body is travelling during the action of the force. Here, Euler made use of this theorem.

[125] Compare [Euler E289, §§ 117–144] and "134. Magnitude ergo harum virium non ex impenetrabilitate quippe quae nullius quantitatis est capax, determinatur, sed ex mutatione status quae effici debet, ne corpora se mutuo penetrent." [Euler E289, § 134]

[126] "Lex. 3. Actioni contrariam semper & aequalem esse reactionem: sive corporum duorum actiones semper esse aequales et in partes contraria dirigi." [Newton, Principia, Axioms]

[127] "29. (...) Ainsi, si deux corps A and B sont pressés l'un contre l'autre par une force $= f$, le corps A repousse le corps B à cause de son impénétrabilité avec une force $= f$, et réciprocement le corps B à cause de son impénétrabilité repousse le corps A avec le même force $= f$." [Euler E181, § 29]

> 31. Cette égalité des forces, d'où dépend le grand principe de l'égalité entre l'action et réaction, est une suite nécessaire de la nature de la pénétration. [Euler E181, § 31]

Euler obtained the new basis by the introduction of *impenetrability* instead of Descartes' notion of extension.[128] Hence, the answer of the question posted in [Euler E015/016, § 102] on the origin of forces in nature is closely related to analytical representation of the changes in velocity. These changes are not independent of each other since the equations are coupled due to the relation between the forces, i.e. $K_{12}+K_{21}=0$, which follows from Newton's 3rd Law (compare Chap. 2). However, the relation between dv_1 and dv_2 is indeterminate as yet since none of the bodies is definitely distinguish from the other by the assignment of parameters. Following Euler, these parameters are to be considered as "internal principles" since the forces belong to the realm of "external principles" or "external causes". The missing notion had to be ascertained by the internal principles in terms of numerical parameters and to ask the question how theses parameters can be specified in the two main experimentally confirmed and theoretically modelled properties of bodies, first the conservation of state, i.e. either the state of rest or the state of uniform motion in a straight direction, for non-interacting bodies and the change of the state for interacting bodies. The answer is that the *internal* properties of all bodies are represented by *inertia* and the magnitude of inertia by the inert *mass* whereas the external principles are analytically represented in terms of the changes of velocities. Hence, Euler joined internal and external principles by a unified analytical representation based on the consequent application of the calculus. The ratio of masses (internally determinate) is expressed in terms of the inverse ratio of change of velocities (externally determinate).

4.1.8 *External Principles: The Increment of Velocity is Independent of Velocity*

Consequences of the subdivision into internal and external principles and conclusions which follow from this distinctions had been summarized by Euler as follows and formulated in terms of analytical representation using differentials:

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[128] "9. *Die erste allgemeine Eigenschaft der Körper bestehet in der Ausdehnung, dergestalt, dass alles, was keine Ausdehnung hat, auch für keinen Körper gehalten werden kann.* Wir sind nicht nur durch die Erfahrung überzeugt, daß alle Körper, so wir kennen, eine *Ausdehnung* haben, sondern unser Begriff von den Körpern schliesset auch die Ausdehnung dergestalt in sich, dass wir vermöge desselben alle Dinge, welche keine Ausdehnung haben, aus der Zahl der Körper mit Recht ausschliessen. Hieran hat nicht nur keine Naturlehre jemals gezweifelt, sondern CARTESIUS (...) ist auch so weit gegangen, dass er das Wesen der Körper in der Ausdehnung gesetzt; wir werden aber weiter unten sehen, dass dieses keineswegs mit Recht geschehen kann; denn obgleich alle Körper ohne Zweifel ausgedehnt sind, so folget, nicht, dass alle Dinge, welche ausgedehnt sind, sogleich zu Körper werden. Ein leerer Raum mag möglich sein oder nicht, so ist doch soviel gewiss, dass der Begriff eines leeren Raumes, welcher unstreitig möglich ist, von dem Begriff der Körper abgesondert werden muss; woraus erhellet, dass unser Begriff von den Körpern noch etwas mehreres als die Ausdehnung allein in sich schliesse." [Euler E842, § 9]

$$\frac{ds}{dt} = \text{const},\ d\frac{ds}{dt} = 0,\ \frac{dds}{dt} = 0,\ \text{homogeneity completed}\ \frac{dds}{dt^2} = 0 \tag{4.18}$$

$$d\frac{ds}{dt} = \frac{dtdds - dsddt}{dt^2} = \frac{dtdds}{dt^2} = dt\frac{dds}{dt^2} = 0 \tag{4.19}$$

$$\text{or}\ \frac{dds}{dt^2} = 0\ \text{since}\ dt \neq 0 \tag{4.20}$$

The advantage of the analytical approach can be demonstrated for the analysis of uniform motion along a straight line since this motion is represented by a scalar function $x = x(t)$ depending on one independent variable t, i.e. the path and the time, respectively. The mechanical model of uniform motion is reduced to the necessary and sufficient geometrical elements, the mass point which is uniformly moving along a straight line. The constant velocity is analytically represented by the relation $ds/dt = v = \text{const}$. Then, additional analytical expressions are solely obtained by the application of the rules of the calculus and, in the next step, are mechanically interpreted[129]: If it holds for a motion along a straight line that $dds/dt^2 = 0$ then this relation is *mechanically* valid since it is a consequence of the inertia of the body [Euler E289, § 105], i.e. the analytical result is in agreement with mechanical principles (and the model of the body independently of the extension or shape of the body). The essential conclusion is that, if $dds/dt^2 \neq 0$ holds for a straight motion, the body cannot only move under the influence of inertia (or cannot only follow the internal principles), but the magnitude has to be assigned to an external cause [Euler E289, § 106].[130] The analytical result is readily generalized for the uniform motion of a body in a *plane* which is related to the axis x and y, $ddx/dt^2 = 0$ and $ddy/dt^2 = 0$ [Euler E289, §§ 107 and 108]. Then, Euler continued to construct the analytical expressions not only in accordance to the *external* principles, but also in conformity to the previously exercised methodology to obtain necessary rules [Euler E015/016, § 152] by logical consistently formulated theorems. For the transition from *internal* to *external* principles, the logical consistency is guaranteed by the procedure and the result is presented in terms of logically consistently formulated analytical expressions. The simultaneous validity of other theorems describing the motion of the same body is excluded, since the only possible versions of the analytical representation are (i) either $dds/dt^2 = 0$ or (ii) $dds/dt^2 \neq 0$. Hence, there is a full logical opposition[131] between *internal* and *external* principles. This procedure is generalized for composite motions which can be considered to be dissolved into or composed of two motions along straight lines which form the axis of a coordinate system in a plane. Hence, there is also in this case:

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[129] Here, Euler made use of the denotation of the "differential quotient" which had been preserved over three centuries until now, but differently interpreted in the 18th and 19th centuries.

[130] To make the conclusion reliable, Eule discussed at first the theorem for "absolute motion" [Euler E289, § 106] before the result is transferred to relative motion (compare Chap. 6).

[131] Compare Schrödinger's analysis who made use of such statements based on rules of logic for the analysis of the relations between classical and quantum mechanics [Schrödinger, Nobel Lecture] (compare Chap. 8).

> 110. Contra igitur, si in quopiam motu, ad directrices OA et OB relato, vel non fuerit $ddx/dt^2 = 0$, vel non $ddy/dt^2 = 0$, vel etiam neutrum, hoc indicio est, corpus non soli inertiae esse relictum, sed ab aliqua actione externa affici. [Euler E289, § 110]

The relations between internal and external principles are analytically expressed in terms of the preservation and change of velocity, respectively. In contrast to the usual approach to distinguish the state only by the velocity, Euler additionally represented the difference in terms of differentials. Hence, the analytical representation is complete. Additional parameters are not necessary.

Internal principles	rest and uniform motion, $v = 0$ and $v = \text{const}$, $dv = 0$
External principles	non-uniform motion, dv is independent of v, $dv \neq 0$
Euler's principle:	The change of velocity is independent of the velocity.

In the next step, the magnitude of the increment $dv \neq 0$ will be considered in dependence on the parameters of the system. The magnitude of the increment of velocity is obtained by joining of internal and external principles:

$$dv = \frac{K_{\text{extern}}}{m_{\text{intern}}} dt. \tag{4.21}$$

Euler assumed that the *increment of velocity is independent of that velocity* the body is moving with before the impact with other bodies. The two problems of mechanics (compare Sect. 2.1) are formulated as follows: (i) Calculate the change in motion and the path of the body if the parameters $K_{\text{extern}}, m_{\text{int}}$ are given and (ii) calculate K_{extern} if the paths are given [Euler E181].[132]

> 131. Apparet haec incrementa celeritatis non pendere ab ipsa celeritate c, sed eundem habitura esse valorem, quantumvis magna aut parva ponatur c. [Euler E015/016, § 131]

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[132] "C'est aussi à quoi aboutissent toutes les recherches de la Mécanique, où l'on s'applique principalement à deux choses: (i) l'une, les forces qui agissent sur une corps étant données, déterminer le changement qui doit être produit dans son mouvement; (ii) l'autre, de trouver les forces mêmes, lorsque les changements, qui arrivent aux corps dans leur état, sont connus." [Euler E181, § 10] The first part had been preferentially demonstrated in the *Mechnica* [Euler E015/016] whereas the second part had been treated later in the treatise *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti* [Euler E065, Additamentum II, § 1]. "Par là on voit qu'il doit y avoir une double méthode de resoudre les problèmes de Mécanique; l'une est la méthode directe, qui est ondée sur les loix de l'équilibre, ou du mouvement; mais l'autre est celle dont je viens de parler, où sachant la formule, qui doit être un *maximum*, ou un *minimum*, la solution se fait par le moyen de la méthode *de maximis et minimis*. La première fournit la solution en déterminant l'effet par les causes efficientes; or l'autre a en vue les causes finales." [Euler E145, § 4.] The starting point of the direct method is an equation of motion [E065, §§ 65–67, Additamentum II, § 6] whereas, in the indirect method, the equation of motion follows from the calculation of the extreme value which is obtained for the integrand of a properly defined integral. Euler demonstrated that the sum of living forces take a minimal value. ("(...) ita ut in curva a corpore proiecto descripta summa omnium viririum vivarum, quae singulis temporis momentis insunt, sit minima" [Euler E065, Additamentum II § 2]). Euler demonstrated that the living forces are invariant for uniform motion [Euler E065, Additamentum II, § 3] and, furthermore, the results are the same as for the direct method in case of motion under the influence of forces for a model system, the trajectories of projectiles [Euler E065, Additamentum II, §§ 4, 5 and 6].

The mechanical limit for the increase of velocity is given by the conservation of momentum. The sum of living forces of a mechanical system is assumed to be of finite magnitude. The interaction between the bodies does neither result in an increase nor a decrease,[133] but always in a redistribution of the living forces. The increase in velocity or the ratio of velocities is limited by ratio of masses. Furthermore, the sum of living forces of a finite non-interacting system is not only conserved, but also limited in magnitude. Any redistribution of living forces among the parts of the system (i.e. the bodies) leads always to a velocity of any of the bodies less than the sum of the living forces of all bodies.

4.1.9 The Proposals of Daniel Bernoulli

The crucial point of the *Mechanica* is, following Euler's argumentation, to answer a question proposed by Daniel Bernoulli in 1726 [Daniel Bernoulli, 1726].[134] Euler wrote that he "treated this question especially thoroughly" [Euler E015/016, § 152] (see above Sect. 4.1.6). Euler did not only reformulate the problem, but aimed to construct a reliable theory resulting in the removal of the *old* theory and the substitution with a *new* theory. The procedure is necessarily accompanied by the substitution of old models with new models. Although the dependence of the change in velocity on the force is assumed to be different, all proposed laws are of the same time dependence. The change in velocity is related to the same infinitesimal increment of time *dt*, but to different powers of forces, i.e. $dc = pdt\ dc = p^2dt$ and $dc = p^3dt$. The mathematical relation between the infinitesimal increment of velocity *dc* and the infinitesimal increment of time *dt* follows from the principles of the *differential calculus* also known by another name as the *calculus of increments* [Euler 1727], [Euler E212] (compare Chaps. 3 and 5).[135] In terms of increments, the change in velocity $\Delta c = \Delta c(t, \Delta t)$ is the function of time t and the increment of the time variable Δt. Generally, the increment of the function is given in terms of a power series $\Delta c = L\Delta t + M\Delta t^2 + N\Delta t^3 + \ldots$ [Euler E212]. In case of infinitesimal increments the increment of velocity is also infinitesimal and, by applying the algorithm for the relations between finite and infinitesimal quantities[136] (compare Chap. 5), the power

[133] "There is neither more nor less potency in the effect as in the cause". "(...) nec plus minusve potentiae in effectu quam in causa contineatur." [Leibniz, Specimen, I (11)]

[134] On year later, Euler began to work in St. Petersburg. Hence, Euler had the opportunity to discuss the problem posted by Daniel Bernoulli (1700–1782) with the author being at that time one of the leading scientists.

[135] Taylor, Brook, *Methodus Incrementorum*, London 1715 (compare Chap. 5).

[136] The algorithm had been already invented by Newton and Leibniz. "(...) sed quia dx vel dy est incomparabiliter minus quam x vel y, etiam dx dy erit incomparabiliter minor quam xdy est ydx, ideoque rejicitur, (...)." [Leibniz, (Gerhardt) p. 377] However the truncation of the power series had not been convincingly justified (compare the criticism by Nieuwentijt, Rolle and Berkeley). The algorithm based on purely numerical relations between finite and infinitesimal quantities or increments is due to Johann Bernoulli and Euler [Johann Bernoulli 1691/92], [Euler 1727], [Euler E212].

series is transformed into $dc = Ldt$ by truncation of all higher order terms. The essential difference between the new representation of the basic laws by Euler and the older representations by Varignon [Varignon, 1700] and Daniel Bernoulli [Daniel Bernoulli, 1726] is due to the explicit appearance of the parameter "mass" in the formula.

$$dc = p \cdot dt (\text{Varignon, Bernoulli}) \quad dc = \frac{p}{A} \cdot dt \quad (\text{Euler}) \tag{4.22}$$

The parameter "mass" denoted by A was not only hidden in the representations of the basic relations presented by Varignon[137] and Daniel Bernoulli, but also in corresponding passages of Newton's *Principia* [Newton, Principia].[138] The reason is that the fall of bodies in the gravitation field does not depend on the mass of the bodies. Assuming the model of empty space,[139] all falling bodies are moving without resistance and arriving after the same time at the surface of the earth independently of the weight and their shape.

The generalization to the 2D and 3D case is straightforwardly performed by an analytical procedure. The only modification consists in completing the relation by those equations which are assigned to other dimensions of the space. In 3D case, the number of additional equation, however, is limited by geometry, but not by the analytical procedure. The mechanical principles are not modified, but are preserved for each direction which is analytically represented by additional independent variables. The latter assumption is a prerequisite for the application of the calculus. Assuming rest and motion beside masses and equilibrium between forces, the paths of bodies which are represented geometrically (a) either by straight lines (b) or curved lines in a plane (c) or curved lines which are not lying in a plane where all the lines are of finite extension. Euler assumed that the common elements of (a), (b) and (c) are of *infinitesimal* magnitude. Therefore, the general principles the theory is based on may be merged and completely represented in three equations related to the three dimensions of the space, i.e.

$$mdu = Pdt, \quad mdv = Qdt, \quad mdw = Rdt \tag{4.23}$$

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[137] "De plus, les espaces parcourus par un corps mû d'une force constante & continuellement appliquée, telle qu'on concoit d'ordinaire la pesanteur, étant en raison composée de cette force & des quarrés des tems employés à les parcourir; l'on aura aussi $ddx = y\,dt\,dt$, ou $y = ddx/(dt\,dt) = dv/dt$." [Varignon 1700]

[138] "Cor. 2. But the errors (...) are as the forces and the squares of the times conjunctly" [Newton, Principia], i.e. $S^{\text{errors}}_{\text{begin}} \sim K \cdot t^2_{\text{begin}}$. This representation is equivalent to $\Delta\Delta s \sim K\Delta t^2$.

[139] Compare Euler's comment on Archimedes [Euler E015/016, § 56]. The *empty space* is a *part of the mechanical model* (compare Section "Euler's world models"). There is an essential difference between the "*geometrical* space" used for the description of motion and the "*empty* space" used for the analysis of the principles of motion (compare Leibniz's early writing on mechanics, "vacuum is an extended non-resisting thing" [Leibniz, Hypothesis] discussed in Chap. 2). Helmholtz described the introduction of an inertial system based on analytical geometry [Helmholtz, Vorlesungen, § 3]. Nowadays, the space (and the time) are characterized as the "background" for physical theories [Smolin].

where P, Q, R are the components of the force and du, dv, dw describe the changes in velocity. Each of the equations represents a motion directed along a straight line. Beside the straight lines, the other basic geometrical elements of the theory are planes where each of the straight lines is oriented perpendicular to a plane. Therefore, geometrically, Euler based the theory on the relation between straight lines and planes. In case of 3D Euclidean space, the three lines correspond to exactly three planes and vice versa.[140]

Geometrically, Euler related motion to straight lines and planes whereas analytically motion is described in terms of *infinitesimal* time intervals. The latter quantities are different from space as well as from points [Euler E015/016]. Thus, the basic rules of the calculus are implemented into a geometrical frame formed by straight lines and planes as basic elements [Euler E177], [Euler E842]. Only in 3D space there is a one-to-one correspondence between the number of lines or axes and the number of planes resulting in the stability of trajectories of moving bodies in a gravitation field [Ehrenfest 1917]. Thus, the complete picture has to be based on coordinates axes and planes where both the geometrical elements have to be interpreted mechanically. The axes are related to the paths and the planes are related to constant values of the potential.

In 1750, Euler transferred the principles developed for the motion of mass points to the motion of extended rigid bodies.

> 20. Soit un corps infiniment petit, ou dont toute la masse soit réunie dans seul point cette masse étant = M; que ce corps ait reçu un mouvement quelconque, et qu'il soit sollicité par des forces quelconques. Pour déterminer le mouvement de ce corps, on n'a qu'à avoir égard à l'éloignement de ce corps d'un plan quelconque fixe et immobile; soit à l'instant présent la distance du corps à ce plan = x; qu'on décompose toutes les forces qui agissent sur le corps, selon directions qui soient ou parallèles au plan, ou perpendiculaires et soit P la force qui résulte de cette composition selon la direction perpendiculaire au plan et qui tachera par conséquent ou à éloigner ou à rapprocher le corps du plan. Après l'élément du tems dt, soit $x+dx$ la distance du corps au plan et prenant cet élément du tems dt pour constant, il sera
>
> $$mddx = \pm P dt^2$$
>
> selon que la force P tend ou à éloigner ou à approcher le corps du plan. Et c'est cette formule seul, qui renferme tous les principes de la Mécanique. [Euler E177, § 20], [Euler E842, § 69]

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[140] In spaces of lower and higher dimension this complementarity is disturbed, there are either more or less geometrical elements of one type. Geometrically, in 3D space there are three and only three straight lines and three and *only* three planes perpendicular to each other. By geometrical reasons, for a number of axes n the number of planes are given by $p = n(n-1)/2$ planes. Only for $n = 3$ is $p = n = 3$ [Ehrenfest]. Therefore, Euler's construction is complete because of the *one plane – one axis* correspondence in 3D space. In contemporary physics, the coordinate system represented by the axes is considered independently of the corresponding complementary picture of planes. However, the adding of additional dimensions changes considerably the relations between axes and planes and, consequently, the mechanical interpretation of the relations described within this extended frame. The most important extension of the 3D space is due to adding *time* as additional variable.

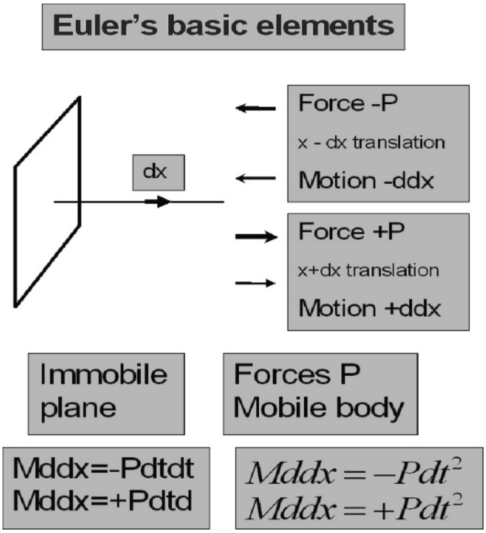

Fig. 4.2 Geometry and forces on Euler' mechanics. The basic elements are formed by planes and straight lines which are oriented perpendicular to the straight lines

Euler implemented the infinitesimal quantities into a geometrical frame where the geometrical relations are independent of the calculus and the foundation of the calculus is independent of the geometrical relations between straight lines and planes.

The basic elements are formed by planes and straight lines which are oriented perpendicular to the straight lines (Fig. 4.2). The directions of forces are geometrically related to planes and mechanically to planes of equal potential.

> 70. Wird ein Körper durch eine Kraft von dieser Fläche gerade fortgestossen, so wird seine Bewegung von dieser Fläche dergestalt zunehmen, dass sich der Zuwachs derselben verhalten wird wie die Kraft multiplicirt mit der Zeit und dividirt durch die Masse des Körpers. [Euler E842, § 70]

The integral $\int Pdx + \int Qdy + \int Rdz$ is called "Wirksamkeit" (compare Chap. 7) and is related to the "living forces".

The development in time is represented by a succession of infinitesimal increments of velocity [Euler E015/016] caused by the presence of forces (where the forces are generated by the interaction of bodies) where the succession of *finite* time intervals

$$v, v + \Delta v, v + 2\Delta v, \&c. \quad \text{with} \qquad \Delta v \sim \Delta t \tag{4.24}$$

is readily transferred into the succession of infinitesimal time intervals

$$v, v + dv, v + 2dv, \&c. \quad \text{with} \qquad dv \sim dt. \tag{4.25}$$

The relations between the representations in terms of finite and infinitesimal quantities are derived in *Institutiones calculi differentials* [Euler E212].

The analytical representation of motion had been invented by Galileo who studied the fall of bodies in the homogeneous gravitation field. Although Galileo did not make use of the idea of gravitation he demonstrated that the bodies are traversing the same distances in the same time independently of their mass. The conclusion is that the observed differences are only due the different resistance caused by the air. Hence, following Archimedes, the general law is formulated for the behaviour of bodies in vacuum [Euler E015/016, § 56]. The vacuum turned out to be an essential part of mechanical models developed for the change of motion by forces. For moving bodies in vacuum, there are no differences between the bodies caused by their different masses. Euler made use of this methodological approach for the formulation of the theorem on the increment of velocity: Generalizing the theorem on the change of rest for the case of motion, Euler postulated[141] that the increment dv is independent of the velocity of the moving bodies.[142] The relation to Galileo's representation discussed in Chap. 1 is readily established (Table 4.2).
Now, it is straightforwardly to model analytically the absence of any resistance by a constant increment of velocity, i.e. by a constant difference between the terms of series (c). Obviously, if the increment of the paths (b) is constant for uniform motion, all terms of series (c) become zero. From Galileo's scheme, it follows that the velocity is defined by the relation between *two* terms of the series or, by one spatial and one temporal interval, whereas the acceleration can only be defined by three terms or, by two consecutive spatial and temporal intervals of the series. As it had been demonstrated by Galileo, the *analytical* procedure is closely related to *experimental* confirmation by measuring the motion in the presence and in the absence of forces (Table. 4.3 and 4.4). Following Euler, Galileo's scheme can be readily interpreted in terms of the *calculus of differences* (compare

Table 4.2 Galileo's representation of the law for falling bodies [Galileo, Discorsi]. Time is represented by finite temporal intervals

Time:	0,		1,		2,		3,		4,…	n, …	(a)
Paths:	0,		1,		4,		9,		16,…	n^2, …	(b)
Diff:		1,		3,		5,		7,…		$2n-1$, …	(c)
Diff:			2,		2,		2,…			$(2n+2-1)-(2n-1)$, …	(d)

[141] Consequently, the opposite statement may be also considered as a reasonable model for the change of motion where the increment of velocity is *not independent* of the velocity or the state of the body (compare Chap. 6). Hence, Euler's interpretation is only valid for the change of rest. Then, the relation $dv = (K/m)\ dt$ can be considered as a necessary truth.

[142] For motion in resisting media, the resistance is assumed to be dependent on the velocity. "367. Lex resistentiae est potestas seu functio celeritatis corporis, cui ipsa resistentia est proportionalis." [Euler E015/016, § 367]

Table 4.3 Galileo's representation of the law for falling bodies [Galileo, Discorsi]. Correlation between temporal and spatial intervals and the increments of velocity

Num	0,	1,	2,	3,	&c.	(a)
Time	t,	$t+\Delta t$,	$t+2\Delta t$,	$t+3\Delta t$,	&c.	(b)
Path	s,	$s+\Delta s$,	$s+4\Delta s$,	$s+9\Delta s$,	&c.	(c)
Veloc.	v,	$v+\Delta v$,	$v+2\Delta v$,	$v+3\Delta v$,	&c.	(d)

Table 4.4 Galileo's representation of the law for falling bodies [Galileo, Discorsi]. Relations between geometric and analytic representation of motion

Mechanics	**Calculus**
Path x	**Series**
$\Downarrow$	$\Downarrow$
Veloc. $\Delta x/\Delta t = v$	1st diff. $\Delta x = v\Delta t + w\Delta t^2 + \dots$
$\Downarrow$	$\Downarrow$
Acceler. $\Delta\Delta x/\Delta t^2 = b$	2nd diff. $\Delta\Delta x = w\Delta t^2 + u\Delta t^3 + \dots$
$\Downarrow$	$\Downarrow$
???	3rd diff. $\Delta^{(3)}x = u\Delta t^3 + p\Delta t^4 \dots$
Uniform mot. $\Delta x/\Delta t = \text{const}$	1st diff. $\Delta x = v\Delta t$
Acceler. $\Delta\Delta x/\Delta t^2 = 0$	2nd diff. $\Delta\Delta x = 0$
Non-unif. $\Delta x/\Delta t \neq \text{const}$	1st diff. $\Delta x = v\Delta t + w\Delta t^2 + \dots$
Unif. acceler. $\Delta\Delta x/\Delta t^2 = \text{const}$	2nd diff. $\Delta\Delta x = w\Delta t^2$
$\Downarrow$	$\Downarrow$
???	4th diff. $\Delta^{(4)}x$
$\Downarrow$	$\Downarrow$
Termination	**No termination**

Chap. 5). The independent variable *time* is represented by an arithmetical series of increments Δt.

The difference between the increment of path in *consecutive time intervals* is only obtained for the combinations of the first and second, the second and third, &c. interval. Otherwise, if there is no difference the motion is uniform, $v = \text{const}$. The independence of the increment of velocity of the velocity is modelled by the uniform acceleration.

The general expression for the change of position or the increment of path in a finite time interval Δt is given by the expression

$$\Delta x = v\Delta t + w\Delta t^2 + \dots \qquad dx = v \cdot dt. \tag{4.26}$$

and the ratio

$$\frac{\Delta x}{\Delta t} = v + w\Delta t + \dots \neq v \qquad ddx = w \cdot dt^2 \tag{4.27}$$

depends on additional terms depending on Δt whereas for infinitesimal time intervals all higher order term vanish (Table 4.4). Hence, in case of $w = 0$ there is no acceleration. Following Newton and Leibniz, Euler introduced two essential improvements, at first the differential calculus is considered as a special case of the calculus of differences [Euler 1727], [Euler E212] and, at second the derived analytical form for this special case in generally valid, i.e. for constant as well as for time dependent velocities, accelerations &c., $v = \text{const}$, $v = v(t)$, $w = \text{const}$, $w = w(t)$, &c., respectively. In the calculus of differences, the quantity Δt is a *variable* (or indeterminate) quantity whereas in the differential calculus the quantity dt is assumed to be determinate or *constant* quantity [Euler 1727], [Euler E212]. Following Nieuwentijt, only differentials up to the second degree, also called differentio-differentials, are considered whereas all higher order differentials vanish [Nieuwentijt, Analysis]. Therefore, the hierarchy of differentials is truncated (Table 4.4, Mechanics). This procedure may be sufficient in the application to mechanics where the acceleration is represented by the differentio-differential of the path. In the general case, however, there is no reason to restrict the mathematical representation of functions (see Table 4.4, Calculus) to those which appropriate for the application in mechanics. The existence of higher order differentials can be traced back to the special mathematical features of the functions. This point of view had been especially stressed by Euler [Euler E212] who consequently removed all restrictions which are solely related to special physical interpretations. On the contrary, Euler paved the way for physical interpretation of the calculus beyond mechanics (compare Chap. 8)

4.1.10 The Operational Definition of Mass

Rest and motion had been analyzed from the very beginning of the investigation of nature. Aristotle introduced the immobile mover, thus, an *asymmetry* between rest and motion had been established. However, motions are *determinate* since all bodies are intent to move towards their natural place. The difference is established between natural motion and forced motion. The problem of relative motion had been considered by Zeno and Aristotle in the model of stadium where three groups of chariots are moving relatively to each other. Later in the 3rd letter to Clarke, Leibniz used a similar model consisting of three groups of bodies to introduce the relational theory of space [Leibniz Clarke] (compare Sect. 2). The whole set up of moving and resting bodies is analyzed from the point of view of an observer who is placed outside the system and who is not taking part in the motion of one the chariots in the Zeno-Aristotle model. This model had been worked nicely until Kepler claimed that the observers themselves are always moving if they are placed at the surface of the earth.

Euler introduced a rigorous and complete *relational theory of motion and forces*. The mass of a body is also defined operationally by the interaction of bodies [Euler E181, §§ 39–42], [Euler E289, §§ 137–144] (on the operational definition of the mass by Euler compare [Jammer, Mass][143] and Couturat [Couturat, Cassirer][144]) in the same manner as it was later proposed by Mach[145] for the relation between masses and accelerations. The relation between the accelerations and the masses

$$m_1 dv_1 + m_2 dv_2 = 0 \tag{4.28}$$

follows from the conservation of momentum, in Euler's derivation from the equality of the magnitude of forces generated in the interaction of bodies which are oriented in opposite directions [Euler E181, §§ 39–42], [Euler E289, §§ 150–55].[146]

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[143] Jammer claimed that even in 19th century "despite the decisive role of this notion there was no "formal definition of mass" [Jammer, Mass, Ch. VIII]. In the introductory part of this chapter, Jammer mentioned that the conservation of mass was implicitly assumed in Newton's *Principia* and the basic assumption on the conservation of mass had been introduced by Kant [Kant, Meta Anfangsgründe] and Lavoisier [Lavoisier, Traité]. Jammer mentioned further authors who are not referring to Euler: M Brisson [Brisson, Dictionnaire], Sigaud de la Fond [de la Fond, Dictionnaire] despite Euler treatise was published in 1736. Jammer stated: "All the textbooks in 18th century do not provide a better definition of mass (as "the mass of the body is the quantity of matter which it contains"). The only exception is Euler's *Mechanica* which had been written aiming a construction of rational mechanics as a science based on axioms, definitions and logical deduction. (...) In the history of the notion of mass Euler's *Mechanica* is of exceptional importance, because it demonstrates the logical transition from the Newtonian axiomatics which is based on the notion of the force of inertia to a widely contemporary notion of mass as a numerical coefficient, which characterizes the special physical body and is determinate by the relation of the force to the acceleration. (...) Euler followed Newton's *Principia*. However, in the demonstration of Proposition 17 (§ 142) appeared a new idea. Euler described the force of inertia (...) as determined by the force which is necessary to force the body to change its state of rest or motion. Different bodies need different forces which are proportional to their quantity of matter [Euler E015/016, § 142]. Therefore, the quantity of matter or the mass is determined using moving forces, an idea, which is of great importance in the following chapters of *Mechanica* as well as in the *Theoria* (1765) [Euler E289, §§ 154–156]. Euler stated that matter (mass) of a body is not measured by its volume, but by the force which is necessary to bring the body in the given motion (acceleration). Here, we find, consequently, the expression of the well known formula 'Force is equal to mass multiplied by acceleration' which expression is used fort he precise definition of mass." Jammer continued with the analysis of the development in France in 19th century, due to Duhamel, Résal, Paul Appell, and other countries, due to Saint-Venant, Reech, Andrade, Kirchhoff, Mach, Hertz and Poincaré, who are used the same definition of mass as Euler (in most cases without reference to Euler). Jammer mentioned that the new concept of mass stimulated also the investigation of the concept of force (compare [Jammer, Force]).

[144] "Pour définir la force vive, c'est-à-dire précisément la quantité qui se conserver dans le choc élastique et qui déterminer par suite la marche ultérieure des mobiles, il fallait tenir compté du facteur *masse*, sans lequel on ne peut l'équivalence des forces vives échangées par le choc. L'invention du concept de masse ne constituait pas seulement un progrès capital de la mécanique: elle permettait à Leibniz de dissocier complètement l'idée de matière de l'idée d'étendue, puisque le coefficient appelé masse est une quantité numérique, et non une grandeur spatiale." [Couturat, Cassirer] [http://facdephilo.univ-lyon3.fr/rev.htm]

[145] "Die wahre Definition der Masse kann nur aus den dynamischen Beziehungen der Körper zueinander abgeleitet werden" [Mach, Mechanik, Chap. 2, § 7, pp. 239–242], [Mach, Mass], i.e. from the relation between the accelerations of at least two bodies.

[146] This relation was common in textbooks on mechancics in 19th centrury (compare [Jammer, Mass, Chap. VIII]). Jammer mentioned Barré de Saint-Venant [Saint-Venant] who used Euler's relation (4.28) (compare [Euler E181. §§ 38–40], published in 1752) almost 100 years later. The

Euler rejected all forces except those which have been called *derivative* forces by Leibniz [Leibniz, Specimen, I (4)]. These forces are generated during the interaction by the interacting bodies [Euler E289, § 131]. They are equal in magnitude and opposite in direction (compare Newton's 3rd Law [Newton, Principia]).

Euler's relational theory of mechanics is crowned by the introduction of observers who are associated with bodies and who are either resting or moving relatively to each other as far as the bodies perform these motions. The action of the observers is the measurement of distances between bodies and the comparison of results of their measurements [Euler E842, §§ 77–83].

Newton defined mass by the product of density and volume. The mass of the body is related to the place (part of the space) the body is occupying.[147] However, Newton introduced also another definition of mass which results from the change in velocity. The ratio of masses is proportional to the inverse ratio of the increments of velocities [Newton, Principia, Axioms]. In the first version, the mass is related to space whereas in the latter version mass is related to motion and the ratio of masses is a numerical factor [Mach, Mechanik]. Following Couturat, Leibniz already made use of the latter representation of mass as a numerical factor in the formulation the law for the conservation of "living forces" [Couturat, Cassirer].

4.2 Extension, Mobility, Steadfastness and Impenetrability

Euler analyzed systematically the basic principles of mechanics in the comprehensive treatise *Anleitung zur Naturlehre* (Instructions for natural science) written between 1746 and 1750 which was only published posthumously in 1862. In the *Anleitung*, Euler gave a systematic and almost complete review of the conceptual problems resulting from the different and in essential parts incompatible approaches in the foundation of mechanics by Descartes, Newton and Leibniz. In the first half of the 18th century, the Cartesian conceptual frame made up of the three substances res infinita sive Deus, res cogitans sive mens and re extensa sive corpus was commonly accepted and still present for the discussion of the basic principles of mechanics although it had been considerably modified by Leibniz and Newton. Especially Descartes postulate on the body as an extended thing had been commonly accepted.

The post-Newtonian scientists were confronted with a legacy rich of new approaches and solutions of old problems, but similarly also rich of open questions. The followers of Descartes, Newton and Leibniz had to struggle with a collection of most complicated problems in mathematics, mechanics and philosophy which

same approach was discussed by Andrade [Andrade] and Mach [Mach, Masse]. Jammer continued the analysis of the development up to 20th century including the *Concept of mass in axiomatic mechanics* [Mach, Mass, Chap. IX].

[147] This definition had been regarded as circular. "Dies ist aber ersichtlich nur eine Scheindefinition, insofern ja die Dichte nicht anderes definiert werden kann als die Menge der Materie in der Volumeneinheit" [Sommerfeld, Mechanik]

were, moreover, closely interrelated from the very beginning. The separation and detachment in different disciplines was only performed in the end of the 18th century and finally established by Kant. However, also in the first half of 18th century the further development was dominated by those who decided not only to reconsider the legacy of all their predecessors without prejudice, but also to made use of all advantages attained by the completion of old methods and the invention of new principles like the principle of least action [Maupertuis, Repos], [Euler E065].

The complete translation of the Leibnizian into Newtonian language was only performed by Euler and d'Alembert. However, a difference remained as far as the status of the basic law of mechanics is concerned (compare Sect. 2). Euler introduced a consequent *relational* theory of forces completed by a *relational* theory of mass discussing the *origin of forces*. The theory is based on a relational theory of motion completed by an early relativistic theory of motion (compare Chap. 6). The Cartesian school was not only influential in France. In Chaps. 1, 2, 3, 4 and 5 of *Anleitung* Euler demonstrated that the body is distinguished from other things like souls and ghosts by the following properties: (i) extension, (ii) mobility, (iii) steadfastness[148] and (iv) impenetrability. The divisibility of bodies is an imperfection of these things [Descartes, Principles, I, § 23].

In the *Anleitung* in contrast to *Mechanica*, Euler reviewed not only all basic problems of mechanics in correlation to the controversially discussed proposals made by different schools, but consitently presented his own suppositions, conclusions and solutions. Euler's paper on *Gedancken von den Elementen der Cörper* [Euler E081] may be regarded as a preliminary version.

4.2.1 Extension and Mobility

Extension and mobility of the bodies had been already discussed before Euler, but it was ever assumed that these properties cannot be treated on an equal footing. Following Euler, extension and mobility are necessary, but not sufficient to distinguish bodies from all other things.[149] Descartes introduced a ranking between the properties by putting the *extension* at the first place as the only necessary and suf-

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[148] Usually, this property of bodies is called "inertia". Euler introduced the German name "Standhaftigkeit" (steadfastness) to describe the behaviour of the bodies which was usually associated with the *inactivity* of bodies and is contrasting with the *activity* and the action of souls and ghosts. The notion of inertia had been introduced by Kepler [Leibniz, Specimen, I (10)] to emphasize indifference wirth respect to rest and motion.

[149] There are extended things which are not bodies [Euler E842, §§ 9 and 10]. A shadow or a picture in the mirror are both extended and mobile, but not bodies. The empty space is also extended, but different from a body.

ficient property of all bodies[150], thus, assigning the priority to extension.[151] Euler introduced another ranking by assigning the priority to the *impenetrability* instead of the *extension*. Euler's construction had to be interpreted on the background of (a) Descartes' notion of the body as a res extensa and (b) Newton's theory of light. In 1650, an alternative theory of light had been developed by Huygens (1629–1695) [Huygens, Traité] derived from the similarity between the propagation of wave at the surface of water.[152] Newton assumed the light consists of small particles which are emitted by the sun. Euler objected to this theory that light ray are penetrating each other without disturbing one the other and developed a theory based on the similarity between the propagation of sound and the propagation of light [Euler E305], [Euler E306], [Euler E307], [Euler E343, Lettres XVI–XXXIII].[153]

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[150] Euler described the general property of all bodies using this precondition. Other things where this property is not observed are excluded from being a body by these purely logical reasons. From this assumption it follows that one can construct other things by observing logical rules that these other things are different from bodies. This construction is independent of the experimental confirmation or any other proof based on experience. "4. *Was allen Körpern ohne einige Ausnahmen zukommt, wird eine Eigenschaft der Körper genannt, und daher werden alle Dinge, in welchen sich diese Eigenschaft nicht findet, von dem Geschlechte der Köper ausgeschlossen*. (...) Wir müssen aber vorher die Eigenschaften aller Körper überhaupt kennen lernen, ehe wir zur Untersuchung der besonderen Eigenschaften, welche nur gewissen Arten von Körpern zukommen, fortschreiten können. (...) Und eben diese Entwicklung der allgemeinen Eigenschaften wird uns nach und nach zu einem vollständigen Begriff der Körper leiten. 5. *Das Wesen der Körper besteht in einer solchen Eigenschaft, welche nicht nur allen Körpern gemein, sondern auch dergestalt eigen ist, dass alle Dinge, welchen diese Eigenschaft zukommt, auch nothwendig für Körper gehalten werden müssen*." [Euler E842, §§ 4 and 5]

[151] Compare the discussion of extension by Châtelet who followed Descartes [Châtelet, Institutions].

[152] "Daher haben auch die spätern Cartesianer die Härte der Kügelchen aufgegeben, und das Fluidum, wodurch das Licht fortgepflanzt wird, elastisch angenommen. Der P. Mallebranche (Mém. de Paris, 1699. p. 32.) setzt an die Stelle der harten Kugeln kleine flüßige Wirbel, deren jeder den empfangenen Eindruck an den nächstliegenden mittheilt. Huygens (Traité de la lumiere, Leiden, 1690. 4.) läßt das Licht so, wie den Schall, aus wellenförmig fortgepflanzten Wirbeln oder Schwingungen eines elastischen Mittels bestehen, und nach Linien fortgehen, welche auf die Reihen der einzelnen neben einander liegenden Wirbel oder ihrer Mittelpunkte senkrecht stehen. Hieraus erweißt er das Gesetz der Brechung, und aus gewissen nicht kreisförmigen, sondern elliptischen Lichtwellen erklärt er die Erscheinungen des Doppelspaths, s. Brechung, Krystall, isländischer. (...) Inzwischen hat sich Herr Euler (Nova theoria lucis et colorum in Opusc. varii argum. Berol. 1746.4. p. 169. seq.) durch die erzählten Schwierigkeiten bewogen gefunden, die von Huygens vorgetragne Hypothese, welche das Licht dem Schalle ähnlich macht (und im Grunde schon ein Gedanke des Aristoteles ist), mit einigen Verbesserungen zu erneuern, und besonders auf die durch Newton sehr erweiterte Lehre von den Farben anzuwenden. Er hat dies mit vielem Scharfsinne und mit Anwendung seiner großen Stärke in mathematischen Berechnungen so glücklich ausgeführt, daß man es noch zur Zeit nicht wagen kan, zwischen seiner Theorie und dem Emanationssystem völlig zu entscheiden." [Gehler, Licht]

[153] After 1740, Maupertuis invented the principle of least action which was mainly developed to obtain a *unified theory of the motion of bodies and the propagation of light*. Since Fermat and Descartes, the main problem was to explain the deviation of propagation of light in case of refraction for inclined incidence. [Maupertuis, Repos] [Maupertuis, Mouvement] [Maupertuis, Accord], [Maupertuis, Examen].

The extension is related to divisibility of bodies since all extended things are divisible and, moreover, divisible in infinity [Euler E842, §§ 11 and 12].[154] Euler summarized the results of the analysis: it is impossible that a body is composed of infinitely many parts.

> Man sieht gemeiniglich diese zwei Sätze *Ein jeder Körper ist ins Unendliche theilbar* und *Ein jeder Körper ist aus unendlich vielen Theilen zusammengesetzt* als gleichgültig an und beweist durch unumstößliche Gründe, dass der letztere unmöglich mit der Wahrheit bestehen könne. Es ist so fern, dass ich diese Gründe wollte zu entkräften suchen, dass ich denselben vielmehr die Kraft eines völligen Beweises beilege und den letzten Satz gänzlich verwerfe. [Euler E842, § 12]

The reason is that the parts of the body are indeterminate if the body is geometrically represented as a continuous thing. This follows from the properties which have been assigned to the continuum.[155] The succession of divisions can be never finished since the parts of an extended thing are also extended. Therefore, the existence of simple least parts is excluded by the assumption that the body can be infinitely divided.[156]

The mobility is discussed as a consequence of the fact that a body is not attached tightly to the *place* it is occupying. The body can be moved to any other place since the body is not bound to a place such that it would be impossible to displace him to any other place [Euler E842, § 16].[157] Euler related the mobility not the *motion* of the body, but to the *displacement* which is performed as an operation *different* from motion. The displacement is only related to the space, it is not related to a certain time. Therefore, the displacement operation does not generate a relation

[154] "Wer demnach dem Körper die Eigenschaften, so mit der Ausdehnung nothwendig verbunden sind, abspricht, derselbe entzieht ihnen die Ausdehnung selbst und schliesst sie folglich von dem Geschlecht der Körper aus." [Euler E842, § 10] In 1736, Euler invented the concept of "bodies of infinitesimal magnitude" [Euler E015/016, § 98] which seems to be in contradiction to the theorem that the extension is a necessary property of all bodies [Euler E842, Chap. 2]. However, the "mass point" is not used as the elementary unit "without extension" which the bodies are composed of, as in the Leibniz-Wolffian theory [Euler E081], but as an appropriate geometrical model for the relation between numbers which have been assigned to geometrical objects like points, lines, surfaces and solids [Euler E842, § 56]. The novelty introduced by Euler is that the parameter "mass" (represented by a number) is not only assigned to an *extended* thing, i.e. exclusively to a "solid or plenum" (having length, depths and breadth [Euler E842, § 15]), but also to "points, lines and surfaces". This procedure is only possible if the "mass" is a purely numerical parameter. Euler generalized the procedure and assumed the *same modus of assignment* for all other mechanical quantities. "Es kommt hier nur daraufan, dass man die verschiedenen Grössen, welche hier vorkommen, als die Kraft, die Zeit, die Geschwindigkeit und die Masse, auf eine bestimmte Art, welche willkührlich ist und auf eines jeden Belieben ankommt, durch Zahlen ausdrücke, (...)." [Euler E842, § 56]

[155] "The continuum, the thing whose parts are indeterminate." [Leibniz, Fragmente]

[156] "13. (...) Die Ausdehnung schliesst nehmlich an und für sich selbst schon alle einfachen Theile aus, weil nach derselben keine letzten Theile zugegeben werden können. 14. *Wann demnach von Theilen eines Körpers die Rede ist, so ist diese Redensart unbestimmt, wofern man nicht hinzusetzt, was für oder die wievielsten Theile des Körper verstanden werden*." [Euler E842, §§ 13 and 14]

[157] "16. Kein Körper ist dergestalt an den Ort, wo er sich befindet, gebunden, dass es nicht möglich sein sollte, denselben an einen jeglichen anderen Ort zu versetzen." [Euler E842, § 16]

between time and space.[158] Then, Euler introduced the difference between rest and motion which had been already discussed in the *Mechanica* [Euler E015/016, §§ 1–6]. A moving body does to stay in any of the places it is traversing [Euler E842, § 19].[159] Then, following the program from 1736, Euler discussed the representation of the moving body by a point [Euler E842, §§ 20–25]. The difference between the extended *space* and the extended *body* is due to the difference in the magnitude of extension, the space is infinitely extended whereas the extension of the body is finite. Additionally, the place as a part of the space has only to be imagined as a part of the space as far as the body is occupying this part. The difference between the places in space is only due to bodies.[160]

> Ohne die Körper würden sich in den verschiedenen Orten des Raumes kein Unterschied befinden, aus welchem man dieselben voneinander unterscheiden könnte; viel weniger wäre es möglich, dass ein Ort auf eine andere Stelle versetzt würde. [Euler E842, § 25]

Euler treated Leibniz's theory as a system of concepts and investigations open for quite different metaphysical, mechanical and methodological interpretations. Euler replaced the Leibnizian monads with bodies and the plenum with a homogenous space being either empty or filled with ether.

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[158] The relation between spatial and temporal intervals is first established by *uniform motion* [Euler E149].

[159] "Die Bewegung ist der Fortgang eines Körper von dem Orte, wo er ist, zu einem anderen". Despite the difference between "translation", "passage" and "Fortgang", the main cofusion results from the replacement of "place which he occupies" with "place where he is". The occupation is not only related to the place, but also to other bodies which are excluded from this place. Furthermore, Euler intended to exclude a "*stay* of a moving" body at any of the places he is occupying while he is moving and described the behaviour of the body by translation (Fortrücken) or passing through a place: "(…) ohne sich an irgedeinem [Orte] auch nur im geringsten aufzuhalten." [Euler E842, § 19] Obviously, Euler argued to exclude Zeno's paradox.

[160] Leibniz claimed that the difference between the parts of the plenum is due to the monads [Leibniz, Monadology, § 8]. Here, Euler transferred the Leibnizian principles from the relation between *monads and plenum* to the relation between *bodies and vacuum* based on a modification of Cartesian principle of extension. Moreover, it becomes obvious that Leibniz's renouncement of the concept of vacuum (which he had accepted in an earlier version "vacuum is an extended thing without resistance", "Corpus est extensum resistens. Vacuum est extensum sine resistentia" [Leibniz, A VI4b1, num. 267]) had serious consequences since it causes the need to ensure the completeness of basic concepts. Following Euler, the notion of body is necessarily completed by the notion of vacuum since both notions are included in the idea of "impenetrability". Instead of inertia, vacuum and impenetrability, Leibniz introduced the *non-inert* and *non-extended* "simple things", called "monads" [Leibniz, Monadology, §§ 1–6]. The monads play an essential role with respect to the *plenum*. "8. Yet the Monads must have some qualities, otherwise they would not even be existing things. And if simple substances did not differ in quality, there would be absolutely no means of perceiving any change in things. For what is in the compound can come only from the simple elements it contains, and the Monads, if they had no qualities, would be indistinguishable from one another, since they do not differ in quantity. Consequently, space being a plenum, each part of space would always receive, in any motion, exactly the equivalent of what it already had, and no one state of things would be discernible from another." [Leibniz, Monadology, § 8]

4.2.2 Uniform Motion: The Division of Time Intervals

The introduction of the uniform motion is the basis for an essential completion of the notions of time and space by quantification. Euler assumed that the uniformity of motion is preserved for *all* parts of time.

> Wenn aber hier von gleichen Wegen, so in gleichen Zeiten durchlaufen werden, die Rede ist, so muss hier solches von allen gleichen, auch von den kleinsten Theilen der Zeit, verstanden werden; es ist nämlich nicht genug, dass alle Stunden gleiche Wege durchlaufen werden, sondern es müssen auch die Wege, so alle Minuten durchlaufen werden, unter sich gleich sein, wie auch diejenigen, so alle Secunden, ja Tertien und so fort durchlaufen werden. [Euler E842, § 22]

In the preceding discussion, Euler excluded the possibility that the body is composed of least parts or particles arguing that it is impossible to obtain such least parts by the division of the body. Now, Euler admitted "smallest particles of time". The assumed procedure does only work without limitation if there are never "smallest parts of a line" or the "path" the point is travelling along. The decisive conclusion is that, in contrast to the subdivision of the extended line (surface, space) the subdivision of the time can be terminated resulting in "smallest parts of time". Here, Euler established an essential difference between space and time. The time interval, e.g. one hour, is divided into *equal* parts. The parts obtained by the division can be further divided into *equal* parts. Therefore, the whole procedure is described by a succession of equivalent operations since the result of any of the operation is always the division of a given time interval into equal parts.

> Dieser Umstand kann am bequemsten nach der Lehre der also Verhältnisse ausgesprochen werden, dass die Wege sich immer wie die Zeiten verhalten müssen. [Euler E842, § 22]

In § 23 Euler introduced the velocity of a uniform motion in a straight direction as the ratio of the path and the time. In § 24 Euler introduced the curved and non-uniform motion and assumed that there is always a uniform motion which coincides at every moment with the curved non-uniform motion. In the next paragraph, Euler discussed the relation between a curved and a straight motion which is mediated by the *temporal instant*. The straight motion can be also *ascribed* (i.e. attributed to, zugeschrieben)[161] temporally, i.e. for an instant, to the curved motion. Therefore, Euler compared *two* motions of *different* type to each other where finally one motion is represented by the other motion, in the present case, the *straight* motion represents in that instant the *curved* motion.[162]

> 24. Ist die Bewegung aber krumlinicht und ungleichförmig, so kann man sich in einem jeglichen Zeitpunkt eine geradlinichte und gleichförmige Bewegung vorstellen, welche in diesem Augenblicke mit derselben übereinkommt; und sowohl die Richtung als die

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[161] Euler did not say that both the motion *coincide* temporarily.

[162] The direction of the curved motion is permanently changing. Therefore, in contrast to the self-sustained uniform motion, it is impossible to adhere rigidly any of these directions to curved motion.

> Geschwindigkeit dieser letzten Bewegung wird auch für diesen Augenblick der ersteren Bewegung zugeschrieben. [Euler E842, § 24][163]

The relation between straight and curved is not universal as far as the direction is concerned, but permanently changing in time and always only valid for an instant. Euler gives an account of this temporal relation between time and instant by means of the geometrical model of the relation between straight and curved.

> 24. (...) Man pflegt zu sagen, eine jegliche Bewegung könne für einen einzigen Augenblick als gradlinicht und gleichförmig angesehen werden, eben wie in der Geometrie die unendlich kleinen Theilchen einer jeglichen krummen Linie mit Recht für grad gehalten werden. Weil aber eine unrichtige Erklärung des unendlich Kleinen leicht Schwierigkeiten machen möchte, so habe ich die Sache auf eine andere Art vorgestellt, (...) hernach wird durch die Differential-Rechnung die Geschwindigkeit gefunden, wenn man das Differentiale des Weges durch das Differentiale der Zeit theilet; eben als wenn die Bewegung durch einen unendlich kleinen Weg gleichförmig wäre. [Euler E842, § 24]

As it had been discussed above, Euler made use of the model for the relation between straight and curved lines, but did not derive the analytical relations from this model. The validity of the mechanical relations are confirmed by their exactness, i.e. Euler demonstrated that the basic relations are obtained from the underlying principles. These relations are necessarily valid, but not only approximately [Euler E015/016, § 152].

4.2.3 Inertia or Steadfastness

Euler referred to Newton's axioms in the chapter entitled *Von der Standhaftigkeit als der dritten allgemeinen Eigenschaft der Körper*. The other two general properties of all bodies, the extension and the mobility, had been discussed before in Chaps. 2 and 3 of the *Anleitung* [Euler E842]. Mobility results in motion, but is also related to rest since a mobile thing is not always impelled to move. The body can also stay at his place without losing the ability to move. Therefore, every body is either resting or moving [Euler E015/016, §§ 1–6]. Rest and motion are considered as *states* of the body. Newton formulated the first and the second axiom for the relation between "body" and an "impressed moving force". Currently, mechanics is based on Newton's axioms as far as the physical foundation is concerned and on the Leibnizian differential calculus as far as the mathematical foundation is concerned. In the 18th century, Lagrange introduced the name "analytical mechanics" (mécanique analytique) [Lagrange, Mécanique]. Newton's axioms represent the physical part whereas the mathematical part is based on geometry. In Chap. 4 of the *Anleitung*, Euler an-

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[163] In this representation, the straight and uniform motion is not a *real* motion, but an conceived motion. Hence, if the *straight* motion is experimentally realized then the *curved* motion is to be necessarily finished and, vice versa, if the *curved* motion is experimentally realized then the *straight* motion is necessarily terminated. This mutual exclusion is described in terms of the relation between uniform and non-uniform motion.

alyzed the axiomatics as far as the physical principles are concerned. Therefore, following Newton, the basic notions are *rest* and *motion* and the concept of *states*.

> 26. Ein Körper, der in Ruhe ist, wird immer in Ruhe verbleiben, wenn er nicht von einer äußeren Ursache in diesem Zustande gestöret und in Bewegung gesetzt wird. [Euler E842, § 26]

The invariance of the state had been explained by the *inertia* of the bodies. This notion had been introduced by Kepler. Newton described *inertia* in terms of a *force of inertia* (compare Chap. 1) whereas Leibniz rejected inertia as basic concept and established a relation between velocity (motion) and forces [Leibniz, Specimen].

The problems appearing in the axiomatic foundation of mechanics can be exemplarily demonstrated for the concept of *inertia* since inertia is closely related to the concept of state of a mechanical system. Assuming only *extension* and *mobility*, the simplest mechanical system consists of one mobile body and all occupied and non-occupied places. Independently of the occupation by the body, the places are assumed to form the space. Following Descartes and Leibniz, the bodies are more than only mobile things, but "extended and resisting" things. The one body – one space model is not appropriate to describe the resistance of bodies, but only the non-resistance of space.[164]

> Corpus est extensum resistens. Extensum est quod habet magnitudinem et situm. Resistens est quod agit[165] in id a quo patitur. Vacuum est extensum sine resistentia. [Leibniz, A VI, 267][166]

Space cannot hinder the body to move, on the contrary, the non-resting space is a precondition for motion. Therefore, resistance can only be defined as a two – body – problem. Following Leibniz, the basic relations which had to be assumed axiomatically are at first (i) the relation between the vacuum,[167] being an "extensum sine resistentia", being *immobile* and the *mobility* or motion of the body, being an "extended mobile thing". Thus, Leibniz assumed indirectly that "space" is an immobile thing whereas the "body" is a mobile thing. The body is not a part of the

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[164] The methodological importance of the vacuum for *statics* (equilibrium) had been already demonstrated by Archimedes who discussed the transfer of the laws for the lever from the empty space into the real world. Euler renewed this procedure and demonstrated the validity of general laws for the *motion* of bodies [Euler E015/016, § 36]. Euler concluded: Following Archimedes, the lever in vacuum will be in equilibrium, therefore, a body in the vacuum will not change its states too, consequently, a body in the real world will also resting until a cause for the change of its state will appear.

[165] Motion can be considered as an "action" and therefore as real, however, "suffering" can never be accounted for real without establishing or release a *connection* to another body.

[166] Following Leibniz, the vacuum is neither acting nor suffering since it is without resistance.

[167] Leibniz correlated bodies and vacuum being *extended* things, i.e. without analyzing the shape of the bodies or the limits of the vacuum. The body is related to a finite part of the space. In the end of the 18th century, the "vacuum" is a notion related to a finite part of the space, an air-pump is called "portable vacuum". "Läßt sich der Teller mit der darauf stehenden Glocke von der Pumpe abschrauben, und der Zutritt der äussern Luft durch einen Hahn unter dem Teller abschneiden, so heißt dies ein tragbares Vacuum (Vacuum portabile)." [Gehler]

space neither the space is a part of the body.[168] At second, (ii) the relation between resisting things since resistance is a relation between two things of the same type (called homogeneous) whereas, logically, mobility is a relation between things to different type. Denoting by B1, B2 and V body1, body2 and vacuum, respectively, it follows:

(a) RES(B1, B2) = RES(B2, B1) called "interaction",
(b) nonRES(B1, V) = nonRES(V, B1) called "mobility" of B1,
(c) nonRES(B2, V) = nonRES(V, B2) called "mobility" of B2,
(d) nonRES(B1, B2) = nonRES(B2, B1) called "absence of interaction",
(e) vacuum is immobile and without resistance, neither mobile nor resisting = not acting.

The relations (a) and (d) are called either "relative motion" or "relative rest" whereas (b) and (c) are related either to "absolute motion" or "absolute rest". From Leibniz's early writings it follows that motion of bodies is either relative motion, either with or without interaction, or relative rest without interaction as far as *resistance* is concerned. In 1695, Leibniz claimed:

> Nec refert, quod omnis corporea actio a motu est, motusque ipse non est nisi a motu sive in corpore jam ante existente sive aliunde impresso. [Leibniz, Specimen, I (1)][169]

In 1698, Leibniz accepted the common interpretation of inertia.

> Quod autem corpora sint per se inertia, verum quidem est, si recte sumas; hactenus scilicet, ut quod semel quiescere aliqua ratione ponitur, se ipsum eatenus in motum concitare non posit, nec sine resistentia ab alio concitari patiatur, non magis quam suapte sponte mutare sibi potest gradum velocitatis aut directionem, quam semel habet, aut pati facile ac sine resistentia, ut ab alio mutetur. [Leibniz, De ipsa, § 11][170]

Then, combining the 1678 and 1698 version, it follows that the bodies only act and suffer if they resist to each other, i.e. in case of motion they are struggling for space free of other bodies to be able to continue their motion unperturbed by other bodies. Although bodies cannot change themselves their state of motion, "non magis quam suapte sponte mutare sibi potest gradum velocitatis aut directionem, quam semel habet", i.e. their motion is always a relative motion [Leibniz, Specimen, I (4), II (2)].

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[168] Using Leibniz's terminology, space and body are "homogonous" (artverwandt), but not "homogeneous" (artgleich) things. "Mobile est homogonum extenso, nam et punctum mobile intelligitur." [Leibniz, Initia]

[169] It does not matter that any corporeal action is from the movement, where the motion itself is not existing unless as from another motion which either did exist in the body before (ante) or is impressed from elsewhere (Newton's impressed moving force). Leibniz described the state before the motion sets in terms of dead forces. This assumption is necessary since only motion can generate another motion [Leibniz, Specimen, I (1)] and, consequently, only a force can generate another force. The dead force is not related to motion, but it is a disposition for motion. These two states are described as *conatus* which results finally in the motion where it is described as *impetus*.

[170] This assumption had been systematically developed by Euler.

Methodologically, Euler recovered the same procedure of demonstrations as it had been used by Archimedes [Euler E015/016, § 56] who considered the equilibrium of the lever in vacuum before he transferred the result into the real world of interacting bodies. Euler related the state of rest to a body which is at absolute rest.

> 56. Corpus absolute quiescens perpetuos in quiete perseverare debet, nisi a causa externa ad motum sollictetur. (...) Ita *Archimedis* demonstratio de aequilibrio bilancis utrinque sibi similis non solum in vacuo, sed etiam, in mundo rei veritatem evincit.
> 57. Est igitur in ipsa rerum natura fundata, quod omne corpus quiescens, nisi ab alia causa externa ad motum sollicitetur, in quiete debeat perseverare. [Euler E015/016, §§ 56–57][171]

This property of bodies as far as "in it lies" ("quantum in se est") had been accentuated by Descartes and Newton. Later, Euler analyzed this property in terms of "internal principles" of motion [Euler E289]. Obviously, the investigation of the "equilibrium of the lever in vacuum" could only be performed as a *thought experiment* by Archimedes. Archimedes and Euler who introduced the model of the lever in vacuum and the model of the mass point, respectively, are important forerunners in the invention of the method which had been later called theoretical physics.[172]

> 60. Simili modo, quo evicimus semel quiescens perpetuo quiescere debere, nisi a causa externa afficiatur, potest ostendi, corpus, quod nunc quiescit absolute, ante hac semper quoque quievisse, siquidem sibi ipsi fuerit relictum.
> 61. Corpus igitur, quod semel quiescit, si ulla causa externa in id neque agat neque egerit, id non solum in posterum quiescet semper, sed etiam ante perpetuo quievisse statuendum est.
> 62. Sequitur ex hoc corpus semel absolute motum in quietem pervenire nunquam posse sibi relictum. Nam si tandem quiesceret, idem oporteret antea quoque semper quievisse, quod est contra hypothesin. [Euler E015/016, §§ 60–62][173]

Based on the principles for rest and using the Archimedean methodology, Euler *transferred* the basic principle of the conservation of state from the state of rest to the state of motion [Euler E015/016, § 62]. This procedure is essentially different from Newton's approach who did not discuss separately at first the laws of rest and did not transfer at second subsequently the results to the case of motion. Moreover, the reliability of the results in ensured since Euler discussed always also the *inverse*

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[171] "56. Ein absolut ruhender Körper wird beständig in Ruhe verharren, wenn er nicht durch eine äussere Ursache zur Bewegung angetrieben wird." [Euler E015/016, § 56]
"57. In der Natur der Dinge ist also das Gesetz begründet, dass jeder ruhende Körper in der Ruhe verharren muss, wenn er nicht durch eine äussere Ursache zur Bewegung angetrieben wird." [Euler E015/016, § 57]

[172] Klein distinguished between *theoretical* and *mathematical* physics [Klein, Elementarmathematik].

[173] "58. Es ist nämlich kein Grund vorhanden, weshalb er eher von der einen, als von der anderen Seite her an seinen Ort gelangt sein sollte und er muss sich daher stets an dem letzteren befunden haben.
61. Ein einmal ruhender Körper, auf welche eine äussere Ursache weder wirkt, noch gewirkt hat, wird nicht nur künftig beständig ruhen, sondern muss sich auch früher stets in Ruhe befunden haben.
62. Ein Körper, welcher sich einmal absolut bewegt, kann nie zur Ruhe gelangen, wenn er sich selbst überlassen bleibt, Käme er nämlich endlich zur Ruhe, so müßte er auch früher stets geruhet haben, was gegen die Voraussetzung ist." [Euler E015/016 (Wolfers), §§ 60–62]

problem. It is demonstrated that also in that case there is no preference of any direction for the arrival at the place the body is presently occupying as it is no preferential direction for leaving the place. In case of motion it follows that a moving body was always moving if it is only on him (quantum in se est). The same principle is valid for rest and motion[174] and, subsequently for absolute and relative motion.

> 81. Quae igitur ex hac corporum natura, quod in statu suo vel quietis vel motus uniformis in rectum permaneant, deducentur, non solum ad motum et quietem absolutam pertinebunt, sed etiam ad eum statum relativum, quo spatium corpusve, ex quo motus aestimatur, uniformiter in directum progreditur.[175]
> 82. Atque etiam non admodum erimus solliciti de motu absoluto, cum iste relativus iisdem contineatur legibus. Et propterea motum hunc relativum ipsum saepius mutabimus in alios huiusmodi, ita tamen, ut traditae leges observentur: si scilicet eum, relatione ad aliud corpus uniformiter in directum quoque progrediens facta, contemplabimur. Qua ratione non cessabit aequabiliter in recta progredi, idque innumerabilibus modis fieri potest, ex quibus, qui commodissimus erit, seligi poterit. [Euler E015/016, §§ 81 and 82][176]

In contrast to Newton, Euler did not assume that the body is provoked to change its motion by an "impressed moving force", but to change its state due to an "external cause" being another body, thereby also changing the state of the other body. Euler did not modify this formulation over the following decades he was working on problems of mechanics. Furthermore, Euler based his proof on a model of the world (and a geometrical properties of the space) consisting of the "infinite and empty space" where none of the possible directions is favoured over any other. There is no sufficient reason (Leibniz's principle of sufficient reason) for a motion since all the directions are *geometrically equivalent* and, therefore, also *mechanically equivalent* since the resting body itself does to move in any of the possible directions.

In the *Mechancia*, referring to Kepler, Euler attempted to get rid of the "force of inertia" [Newton, Principia, Definitions], but did not succeed in replacing this notion by an appropriate alternative supposition on the properties of bodies. The alternative approach had been finally related to the impenetrability of bodies (compare next section).

> 74. *Vis inertiae est illa in omnibus corporibus insita facultas vel in quiete permanendi vel motum uniformiter in directum continuandi*.

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[174] The relations between "absolute motion" and "relative motion" had been carefully analyzed by Euler (compare [Euler E015/016, §§ 81–92]) who finally rejected absolute motion [Euler E842]. Euler's theory of relative motion had been only rediscovered in the end of the 20th century.

[175] "81. Was wir daher aus der Eigenschaft der Körper, den Zustand der Ruhe oder der gleichförmig geradlinigen Bewegung beizubehalten, ableiten, bezieht sich alles nicht bloss auf die absolute Bewegung und Ruhe, sondern auch auf den relativen Zustand, in welchem der Raum und der Körper gleichförmig in gerader Linie fortschreiten." [Euler E015/016 (Wolfers), § 81]

[176] "82. Wir werden uns demnach wenig um die absolute Bewegung kümmern, da jene relative in denselben Gesetzen enthalten ist. Wir werden oft diese relative Bewegung in andere derselben Art verändern, jedoch so, dass die aufgestellten Gesetze beobachtet werden; wenn wir sie etwa mit Bezug auf einen anderen, ebenfalls gleichförmig und geradlinig fortschreitenden Körper betrachten. In diesem Falle wird er nicht aufhören, gleichförmig und geradlinig fortzuschreiten und dieses kann auf unzählige Weise geschehen, wovon man die bequemste auswählen wird." [Euler E015/016 (Wolfers), §§ 81 and 82] This result had been never questioned and also later preserved [Euler E842], [Euler E177].

> 76. *Keplerus*, qui primus hanc vocem formavit, tribuit eam ei vi, quam omnia habent corpora, resistendi omni illi, quod ea de statu suo deturbare conatur; atque haec vox inertiae melius cum resistentia ideae congruit quam illa perseverantiae, cum qua nos coniunximus. Sed facile intelligiur has definitiones re a se invicem non differre; eadem enim est vis motum vel quietem continuans, et quae impedimentis resistit. Malui vero hac uti definitione quam *Kepleriana*. [Euler E015/016, §§ 74 and 76][177]

Here, Euler distinguished between the forces and the property or the ability of the bodies causing these forces.[178] Later, Euler attributed this ability to the impenetrability of bodies which causes this force. The inertia is different from a force since it will be demonstrated that neither rest nor uniform motion is caused by forces [Euler E842]. Nevertheless, Euler made use of the commonly accepted interpretation in 1736.

> 189. *Quando potentiae et motus directiones in eadem sitae sunt recta, motus erit rectilineus*. Omne corpus vi insita conatur motum suum in directum continuare, id quod semper praestat, nisi impediatur (§ 65). [Euler E015/016, § 189][179]

In the 1740's years, Euler removed the ambiguities resulting from the notion of force of inertia and replaced completely the previously questioned notion of the "force of inertia" by the supposition that the forces are only generated in the interaction of bodies due to the impenetrability.

4.2.4 Impenetrability, Inertia and Forces

In the end of the 1740's years, impenetrability becomes the basic notion in Euler's mechanics elaborated in the paper *Recherches sur l'origine des forces* [Euler E181] published in 1750 and in the comprehensive treatise *Anleitung zur Naturlehre* [Euler E842] published only posthumously in 1862. The impenetrability is just that

[177] "74. Die Kraft der Trägheit ist jene allen Körpern innewohnende Fähigkeit, entweder in Ruhe zu verharren, oder ihre Bewegung gleichförmig in gerader Linie fortzusetzen. 75. Diese von der Natur der Körper abhängende Ursache des Verharrens in seinem Zustande ist die so genannte Kraft der Trägheit. 76. *Kepler* hat zuerst diese Benennung aufgestellt und er bezeichnete damit die jedem Körper eigenthümliche Kraft, allem zu widerstehen (resisting), was seinen Zustand zu verändern bestrebt. Zwar passt das Wort Trägheit besser zu der Idee des Widerstandes, als zu der obigen des Beharrens, womit wir dasselbe verbunden haben, allein in der Wirklichkeit sind diese beiden Erklärungen nicht voneinander verschieden. Es ist nämlich dieselbe Kraft, welche die Bewegung oder Ruhe fortsetzt und welche Hindernissen widersteht. Ich wollte lieber diese als *Kepler*'s Erklärung anwenden, weil man noch nicht weiss, auf welche Weise die Körper den antreibenden Kräften widerstehen. Ausserdem hat diese Kraft des Widerstandes ihren Ursprung in dem Vermögen, die Ruhe und Bewegung fortzusetzen und muss daher hieraus ableitet werden." [Euler E015/016 (Wolfers), §§ 74–76]

[178] "The origin of this force is the ability of the body to preserve (to continue to stay in) the state of rest and motion and, therefore, it is to be derived from this ability." [Euler E015/016, § 76] This ability had been later explained as being caused by impenetrability.

[179] "189. Jeder Körper hat durch eine ihm innewohnende Kraft das Bestreben, seine geradlinige Bewegung fortzusetzen und wird diess immer thun, wenn er nicht daran gehindert wird (§ 65)." [Euler E015/016 (Wolfers), § 189]

property of the bodies making them different from all other things. An impenetrable thing is necessarily a body. Therefore, the impenetrability is necessary and sufficient to determine all basic properties of bodies.[180]

Supposing that the conservation of state is *independent of forces*, but is explained by *inertia*, Euler transformed the idea of equilibrium into the idea of the conservation of state *due to inertia*. The new approach is obtained from the consequent and justified distinction between *internal* and *external* principles whose validity is guaranteed by inertia and forces generated by the interacting bodies. There are no forces at all being assignable to non-interacting bodies[181] whereas the inertia is not only not qualitatively modified by the interaction, but is also quantitatively preserved in magnitude.

> 8. Toute cause qui est capable de changer l'état d'un corps s'appelle *force* et partant, lorsque l'état d'un corps change, soit que du repos il commence à se mouvoir, ou qu'étant déjà en mouvement, il change ou de vitesse ou de direction, ce changement vient toujours d'une force et cette force se trouve hors du corps dans quelque autre sujet, quel qu'il soit.
> 32. Nous voyons donc que la seule impénétrabilité des corps est capable de fournir des forces, par lesquelles l'état des corps peut être change; et dans les cas où cela arrive, si l'on demande, d'où viennent les forces qui causent ces changemens, on pourra répondre hardiment que l'impénétrabilité des corps en est la véritable source. [Euler E181, §§ 8 and 32]

As a consequence, the subject of natural science is to study the changes which happen in the world of bodies. [Euler E842, § 1] It is impossible that two bodies can simultaneously occupy the same place.

> 35. Ein jeglicher Körper muss in dem Raume einen besonderen Ort einnehmen, und es ist unmöglich, dass zwei Körper zugleich an eben demselben Orte sein könnten. [Euler E842, § 35]

Following Euler, the notion of impenetrability included the notions of extension, mobility and steadfastness (inertia). Although Descartes assumed that the bodies are characterized by *extension* he thought that the impenetrability was included and connected to extension.

> 38. *Die Undurchdringlichkeit schliesset für sich schon die Ausdehnung und Beweglichkeit und folglich auch die Standhaftigkeit in sich. Wenn man also dem Körper die Undurchdringlichkeit zueignet, so muss man ihm auch die übrigen Eigenschaften zuschreiben*. (...) Diese Eigenschaft wird auch von allen denjenigen, welche von der Natur der Körper geschrieben, ohne einige Ausnahme zugegeben, und ungeachtet CARTESIUS das Wesen der Körper in der blossen Ausdehnung gesetzt, so hat er doch geglaubt, dass die Undurchdringlichkeit mit der Ausdehnung verbunden sei. [Euler E842, § 38]

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[180] Also Leibniz could not get rid of inertia. "Quod autem corpora sint per se inertia, verum quidem est, si recte sumas; (...)." [Leibniz, De ipsa, (11)] Châtelet joined extension, active and passive forces: (i) Extension (Descartes), (ii) active force (Leibniz) and (iii) passive force (Newton, force of inertia) [Châtelet, Institutions].

[181] For the interpretation of interaction of bodies due to gravitation compare *Anleitung* [Euler E842, Chap. 19].

Consequently, the changes in the world can be distinguished into those which are free of any influence or intervention by ghosts. These changes are solely related to the forces caused by impenetrability.

> 49. Alle Veränderungen, welche in der Welt an den Körpern vorgehen, insofern dazu von Geistern nichts beigetragen wird, werden von den Kräften der Undurchdringlichkeit der Körper hervorgebracht, und finden also in den Körpern keine anderen als diese Kräfte statt. [Euler E842, § 49]

Hence, Euler rejected (i) Descartes' supposition that the direction of motion can be modified by ghosts,[182] (ii) Malebranche's assumption of occasional intervention by God and, finally (iii) Leibniz's system of prestablished harmony. Leibniz's construction is not indispensable. However, as it will be demonstrated, Euler made use Leibniz's construction of *observers* who attained different pictures or perspectives of the world[183] due to their connection to the bodies as the only frames of reference for relative motion (compare Chap. 6). Here, Leibniz's relational approach to time and space had been successfully developed by Euler.

Euler exclusively described the interaction of bodies in terms of the (i) preservation and (ii) change of their states without any reference to the actions of an observer who is modifying the behaviour of the bodies [Euler E842, § 49]. The observer is *not absent*. On the contrary, he is a part of the world or the mechanical system, but his *influence* upon the *world of bodies* had been excluded. As it had been discussed above, Euler rejected Descartes', Malebranche's and Leibniz's construction of body–soul relations. The actions of the observers are by *no means* in *harmony* with the actions of bodies since the bodies are changing mutually their states by forming a whole ("coeunt" [Euler E289, § 131][184]) while the observers are at the most interchanging informations without any mutual change of their states. Moreover, the observers are not related to one of the interacting bodies since, following Euler, their "labs" consists of other uniformly moving or non-interacting bodies. Hence, the setup presented by Euler is to be completed by the analytical representation of uniform motion (compare Chap. 6).

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[182] Here, Euler is in agreement with Leibniz. However, Euler's conclusions are quite different. Leibniz stated: "80. Descartes recognized that souls cannot impart any force to bodies, because there is always the same quantity of force in matter. Nevertheless he was of opinion that the soul could change the direction of bodies. But that is because in his time it was not known that there is a law of nature which affirms also the conservation of the same total direction in matter. Had Descartes noticed this he would have come upon my system of pre-established harmony." [Leibniz, Monadology, § 80]

[183] "57. And as the same town, looked at from various sides, appears quite different and becomes as it were numerous in aspects [perspectivement]; even so, as a result of the infinite number of simple substances, it is as if there were so many different universes, which, nevertheless are nothing but aspects [perspectives] of a single universe, according to the special point of view of each Monad." [Leibniz, Monadology, § 57]

[184] "Si duo corpora ita coeunt, ut neutrum statum suum conservare possit, quin per alterum penetret, tunc in se mutuo agunt viresque exerunt, quibus eorum status mutetur." [Euler E289, § 131]

4.2.5 Summary: Euler's Axiomatics

Euler developed systematically the foundation of mechanics in the treatises *Mechanica* [Euler E015/016] 1736, *Anleitung zur Naturlehre* 1746–1750, *Origin of forces* [Euler E181] 1750 and *Theoria* [Euler E289] 1765. The basic concepts are rest, motion, mass and forces. The summary of principles had been given by Euler in the introductory part of *Theoria* [Euler E289, §§ 1–259]. According to the program from 1736 [Euler E015/016, § 98] (compare Sect. 4.1), the basic principles of the motion of mass points remain to be valid also for the theory of motion of extended bodies (Table 4.5). Therefore, the first part of *Theory of motion of rigid bodies* is entitled "Introduction comprises necessary explanations and additions to the motion of points" [Euler E289, §§ 1–259]
Euler preserved the distinction between absolute and relative motion. The frame of references is established by a point uniformly moving in straight direction [Euler E289, § 238].

The editor of the *Mechancica* summarized the progress Euler made by saying that Euler invented a "completely new approach of the theoretical representation of mechanics being also different from Newton's axiomatics in the *Principia*" [Euler, Opera omnia, II, 1, (Stäckel)]. Euler intended not only to replace the geometrical methods by analytical representations, but surpassed all his famous and distinguished predecessors and contemporaries.

> Die Versuche, die *Newton* und *Johann Bernoulli, Varignon* und *Hermann* nach dieser Richtung hin gemacht hatten, stehen weit zurück hinter der Neugestaltung, die wir Euler verdanken, und die Mechanica bleibt, wie es Lagrange in seiner *Mécanique analytique* (1788) ausgesprochen hat, 'le premier grand ouvrage où l'Analyse ait été appliquée à la science du mouvement'. [Euler, Opera omnia, II, 1, (Stäckel)]

The reason for this progress is that Euler did not confine the reinterpretation and reconsideration to mechanics, but completed his efforts by performing the same procedure in mathematics. Employing the denomination of "arithmetization" introduced later by Felix Klein to describe the development of mathematics in the end of the 19th century [Klein, Arithmetization], Euler's approach should to be adequately described in its properly meaning only as a *simultaneous arithmetization* of *mathematics and mechanics*. Hence, Euler paved not only the way for the later development of mathematics in the 19th century which had been completed by Weierstraß

Table 4.5 Euler's axiomatics represented by the table of contents of the *Theoria* [Euler E289]

Introductio continens illustrationes et additiones necessarias de motu punctorum	
Cap. I.	Consideratio motus in genere
Cap. II.	De internis motus principiis
Cap. III.	De causis motus externis seu viribus
Cap. IV.	De mensuris absolutis ex lapsu gravium petitis
Cap. V.	De motu absoluto corpusculorum a viribus quibuscunque actorum
Cap. VI.	De motu respectivo corpusculorum, a viribus quibuscunque sollicitatorum

[Klein, Arithmetization],[185] but anticipated also essential developments of the 20th century physics introduced by Einstein [Einstein, Bewegte], [Einstein, Allgemeine Relat] (compare Chap. 6).

4.3 Euler and His Contemporaries

In 1727, Euler started his scientific and, stimulated by the Bernoulli brothers Johann and Daniel he settled in St. Petersburg. This year marked the beginning of the post-Newtonian period which may be regarded simultaneously also a post-Leibnizian period. The debates between Newton and Leibniz were already truncated by the death of Leibniz in 1716. Nevertheless, the debates had been also continued after the death of Newton in 1727. The legacy of Descartes, Newton and Leibniz had been reconsidered especially by those scientists who belonged to the new generation born in the beginning of the 18th century. Between the years 1734 and 1756, Maupertuis (*1698), Daniel Bernoulli (*1700), Châtelet (*1706), Euler (*1707) and d'Alembert (*1717) published important writings on all controversial topics treated before by Newton and Leibniz.[186] Finally, Euler achieved the breakthrough by generalizing and unifying mechanics and mathematics. This will be discussed for the theory of relative motion and the measures of the quantity of motion and living forces. However, the comprehensive summary of this unification entitled *Anleitung zur Naturlehre* was only published posthumously in 1862. Although the *Anleitung* was written in German and had been published in 1862 and, moreover, Euler's *Mechanica* and *Theoria* had been translated into German by Wolfers[187] in 1848 and 1853 the impact on German writing scholars like Mach was negligible (compare [Mach, Mechanik]). Nevertheless, the development of mechanics in the post-Newtonian period in the 18th century and later had been successfully continued by Lambert (1728–1777), Lagrange (1736–1812), Lichtenberg (1742–1799), Laplace (1749–1827), Jacobi (1804–1851), Hamilton (1805–1865), Helmholtz (1821–1894), Maxwell (1831–1879), Lorentz (1853–1928), Poincaré (1854–1912) and Minkowski (1864–1909).

Nowadays, the complicated development had been condensed into a short introduction of the textbooks and the original versions and intermediate results are treated as fossile records of a time long ago. Nevertheless, there is an interesting renewing and revival of demand for the original versions and contribution by the predecessors. In the end of the 20th century, Chandrasekhar translated Newton's *Prin-*

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[185] "Ich möchte zuerst darauf hinweisen, daß das von Taylor zwischen Differenzen- und Differentialrechnung geknüpfte Band noch lange Zeit gehalten hat: Noch in den analytischen Entwicklungen Eulers gehen beide Disziplinen Hand in Hand." [Klein, Elementarmathematik, p. 253]

[186] It may be useful to bring to mind that life and scientific activity of Johann Bernoulli (1667–1748), Ch. Wolff (1679–1754), Voltaire (1694–1778), Maupertuis (1698–1759) and Daniel Bernoulli (1700–1782) are closely related to the period between 1730 and 1755 we analyze here when Euler, Châtelet and d'Alembert started their scientific career.

[187] Wolfers translated also Newton's *Principia* into German (1872).

cipia into the Leibnizian language of the calculus undertaking the same task of doing which had initially demonstrated by Euler between 1734 and 1736. In that time, Euler performed complete translation of the Newtoninan into the Leibnizian language completed by the complementary translation of the Leibnizian into Newtonian language. Following Euler, d'Alembert and Châtelet similarly proceeded thereby modifying to some extent the terminology (compare the translation of the *Principia* into French by Châtelet), but preserving mainly the basic mechanical principles. It is the merit of Euler, Châtelet and d'Alembert to have discovered and demonstrated that there is a *hidden* common basis of both theories. Obviously, the famous authors stressed and overrepresented the differences in their approaches, but screened carefully and consciously this common basis. This complicated, but nevertheless very interesting relation and relationship may be demonstrated in case of the invention of the calculus. After a long-lasting debate and numerous subsequent investigations of the details of the debate summarized by the title of the book "priority and equivalence" [Meli] the only conclusion can be that only Leibniz was able to understand immediately Newton's cryptography as, in the opposite case, only Newton was able to acknowledge immediately the progress Leibniz made. Therefore, Newton and Leibniz *agreed* almost perfectly in the meaning and weighting of the *basic* questions and problems as far as the legacy of *Archimedes, Galileo* and *Descartes* was concerned, but represented their developments and steps beyond their predecessors within a frame of notions and in languages which turned out to be outwardly quite *different* from each other. Guided by mathematical principles and seeking for logical consistency of the theory, Euler discovered this common basis. For Euler, the veritable reliability of mechanics is attained if the principles of mechanics are "de pair avec les vérités géométriques" [Euler E343, Lettre LXXI], i.e. if the statements are not only true, but "necessarily true" [Euler E015/016, § 152].[188]

In 1740, Châtelet repeated the procedure performed by Leibniz in 1695 and 1698 and translated Newton's basic assumptions into the Leibnizian terminology. The *derivative* forces result from the *impact of bodies* which is, however, explained by the *conflict* of the *primitive* or elementary forces. Following Euler, as it had been discussed above, the origin is not due to *inherent forces*, but due to impenetrability.

> Il y a deux sortes de force motrice[189]; M. de Leibnits appelle la force qui se trouve dans tous les Corps, & dont raison est dans les Élémens, *force primitive*, & celle qui tombe sous nos sens, & qui naît dans le choq des Corps, du conflict de toutes les forces primitives des Élémens, *force derivative*; cette derniére force découle de la première, & n'est qu'un Phénomene, comme je vous l'ai expliqué plus haut. [Châtelet, Institutions, § 158]

In contrast to Newton, who assumed a *relation* between phenomena and forces, Châtelet considered the relation between forces. Even without any experimental

[188] "Apparet igitur non solum verum esse hoc theorema, sed etiam necessario verum." [Euler E015/016, § 152] Here, Euler adopted Leibniz's methodology [Leibniz, Monadology, §§ 31–39] based on the *principle of contradiction*, the *principle of sufficient reason* and the distinction between *contingent* and *necessary truths* [Euler E081], [Euler E842].

[189] Leibniz did not call this force "moving" force. In contrast to Newton, who assumed *impressed moving* forces Châtelet called these forces either moving or impressed forces.

equipment we can qualitatively distinguish between rest and motion of a body relatively to our position. The equipment allows for quantification, i.e. for measuring the distance the body is travelling. The presence of a force is only indicated experimentally by the *change* of the motion. The force cannot be considered as a phenomenon as the motion is. Therefore, following Euler, we have to answer the question how the phenomena can be described first in the *absence* and, second in the *presence* of forces. Euler assumed that the phenomena are represented by the paths of bodies and distinguished between three types of paths or three types of motions [Euler E289, § 22] whereas Châtelet distinguished between different types of forces.

> 229. La force active & la force passive des Corps, se modifie dans leur choq,[190] selon de certaines Loix que l'on peut réduire à trois principes. [Châtelet, Institutions, § 229][191]

In contrast to Euler who introduced later a general and fundamental law for mechanics which is valid for rest and motion [Euler E177], Châtelet treated Newton's Laws as "Lois générales du mouvement", i.e. preferentially for motion [Châtelet, Institutions, § 229]. Moreover, Châtelet interpreted Newton's 2nd Law in terms of Leibniz's principle of sufficient reason. Châtelet's translation of Newton's *Principia* demonstrates this modification.

> First Law: Un Corps persévére dans l'état où il se trouve, soit de repos, soit de mouvement, à moins que quelque cause[192] ne le tire de son mouvement, ou de son repos.[193]
>
> Second Law: Le changement qui arrrive dans le mouvement d'un Corps, est toujours proportionnel à la force motrice qui agit sur lui[194]; & il ne peut arriver aucun changement dans la vîtesse, & la direction du Corps en mouvement que par une force extérieure; car sans cela ce changement se seroit sans raison suffisante.
>
> Third Law: Le réaction est toujours égale à l'action; car un Corps ne pourroit agir sur un autre Corps, si cet autre Corps ne lui resistoit; ainsi, l'action & la reaction sont toujours égales & opposées. [Châtelet, Institutions]

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[190] Following Châtelet, the active and the passive forces are *modified* due to the impact of bodies whereas Euler claimed that the forces are *generated* in the impact by the bodies.

[191] In the German translation, a part of the paragraph is not translated. "Die thätige und die leidende Kraft der Körper wird in ihrem Stoße (die im Stoße *modifiziert* werden) nach gewissen Gesetzen eingerichtet, welchen man hauptsächlich auf drey bringen kann." [Châtelet, Naturlehre, § 229] Here, Châtelet recovered the early version of the dynamics in Newton's treatise *De motu*. "Letztlich konzentrierte sich also Newtons Dynamik weiter auf das Zusammenspiel von inhärenter und eingedrückter Kraft." [Westfall, Newton, p. 216]

[192] In the *Principia*, Newton used the word *force*. "A body preserveres in its state of rest or uniform motion in a right line, unless it is compelled to change that state by a forces impressed thereon." [Newton, Principia, Axioms] In the translation, Châtelet replaced the word *force* with the word *cause*. Châtelet's formulation coincides with Euler's formulation of the basic law. It should be stressed that Euler put the law for rest at the first place, the law for motion at the second. Furthermore, Euler chose the same grammatical and logical structure stressing the existence of the same *algorithm* which is used for expressing the relation between the basic concepts. From this representation it can be concluded that there is *one and the same law* which governs the *change of the state* of a body independently of the difference between rest and motion [Euler E015/016]. The "external cause" for the change of the state is the presence of another body [Euler E181], [Euler E842].

[193] The first Law is compatible with Euler's as well as Châtelet's interpretation of the different types of forces.

[194] Here, Châtelet omitted the word "impressed".

In the 3rd Law Châtelet recovered the original version of Newton's law for the motion resulting from an action composed of inherent and impressed forces (compare [Westfall, Newton, p. 216]). However, later in the *Principia*, Newton did not specify the forces into "active" and "resisting" forces, but stressed that they appeared in the mutual actions of two bodies.

> To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. [Newton, Principia]

Châtelet combined notions of different origin. This procedure follows from Châtelet's assumption to reinterpret the basic laws in terms of *extension, active* and *passive forces*. In agreement with Euler's assumptions, Châtelet decoupled the relation between the preservation of state and the absence of "impressed" forces replacing the terminus "force" by "cause". The word "impressed" is also omitted in the translation of the second law and is replaced with "external forces". Châtelet added the explanation that "otherwise the change would be without sufficient reason". An "external" force should be related to an "internal" force. "Active" and "passive" forces are either both internal forces, being "abilities" of the bodies, or neither internal nor external forces. From this symmetry of action and reaction, Euler concluded that the forces are generated by the bodies.

The main difference between Euler's and Châtelets approach is that Châtelet discussed the basic notions of mechanics independently of their representation in the frame of the calculus. Therefore, the correlation is mainly established between *metaphysical* and *mechanical* notions instead of the relation between *mathematical* and *mechanical* principles. As a consequence, the metaphysical principles cannot be convincingly rejected nor the mechanical principles can be sufficiently confirmed. Following Euler, all metaphysical, methodological and mechanical principles are to be proved whether they can be reasonable joined with mathematical representation by the calculus. The mechanical principles cannot be *confirmed* by the calculus, but can be checked for their *compatibility* with the rules of the calculus. As Euler demonstrated, one and the same force cannot simultaneous *preserve* and *change* the state of the body whereas one and the same force may be cause a change of the state which is either represented by the change of the momentum or the change of living forces, i.e. either $mdv = Kdt$ or $mvdv = Kds$ [Euler E015/016, §§ 130–155, 213] (see below).

Why to read Euler in the 21st century? Why to read Châtelet? The surprisingly short, but nevertheless precise answer is that Euler, Châtelet and their contemporaries were involved in the same business we are dealing with still today. The question to be answered in the 18th century and today concerns the legacy of Descartes, Newton and Leibniz. Nowadays, it has to be completed by the inclusion of the legacy of Euler, Châtelet and their contemporaries.[195] As far as Leibniz's work is concerned it is obvious that essential parts had been published only in the 19th and 20th centuries and even today the edition is not finished, but still progressing. Having read the unpublished Leibnizian manuscripts

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[195] For the present state of affairs compare *Leonhard Euler. Life, work and legacy*, ed. by Robert E. Bradley and C. Edward Sandifer, Elsevier 2007 [Bradley, Sandifer].

on logic Russell (1872–1970) and Couturat (1868–1914) developed a basic idea [Russell] which was completely different from the version introduced by Mach (1838–1916) at the same time [Mach, Mechanik] who considered Leibniz preferentially as philosopher and theologian whose theory of the best of all possible worlds "hat durch Voltaire die gebührende Abfuhr erhalten" [Mach, Mechanik].[196] Although Helmholtz (1821–1894) credited Leibniz's assumption on the conservation of living forces [Helmholtz, Vorlesungen] Mach's interpretation was later commonly accepted in the 20th century.[197]

Mach's interpretation was in the track of d'Alembert (1717–1783) who stressed in 1743 that the debate was not of interest for mechanics, but preferentially metaphysically motivated. However, later in 1766, Kästner (1719–1800) claimed although he agreed with d'Alembert as far as the "battle of words" is concerned that the debate has stimulated the development of mechanics.[198]

> Dieser Streit ist also, ihm den rechten Namen aus der griechische Grundsprache zu geben, eine Logomachie gewesen; doch ist selbst ein solcher Wortstreit, in der Mathematik lehrreicher gewesen; als Wortstreite in anderen Theilen der Gelehrsamkeit zu seyn pflegen. Denn die vielfältigen Untersuchungen, Auflösungen und Aufgaben u.d.g. die er veranlaßt hat, haben doch unsere Kenntniß wirklich erweitert. [Kästner, Anfangsgründe]

Therefore, reading Châtelet and Kästner today we obtain a lively mirroring of the influence of Newton and Leibniz and, moreover, also of Euler in the first half of the 18th century which is not obscured by later interpretations.[199] First of all, we learn that Leibniz and Newton invented almost simultaneously not only the calculus, but also basic concepts of mechanics in 1686 and 1687, respectively.

What are the remaining questions which had not been treated until a commonly accepted solution was obtained? There are two main problems: (i) the measures of motion and forces and (ii) the relativity of motion. The correlation between (i) and (ii) had been already stressed by Euler in 1736. Euler's foundation of mechanics was appropriate to terminate the debate already in 1736 since Euler demonstrated[200] that there are not *different forces*, but *different effects* of *one* and the *same force* resulting

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196 Obviously, Mach was not aware of the *mathematical* background, the max-min principle, introduced by Leibniz which was later elaborated by Euler, Lagrange and Hamilton. The mathematical relations are *independent* of the metaphysical and the mechanical interpretation. The metaphysical assumption is that a mechanical interpretation is possible, but the mathematical properties and relations are *independent* of metaphysics, i.e. the metaphysical rules had to be *added*. Therefore, Leibniz claimed: "cujus leges metaphysicas extensionis legibus addendo" [Leibniz, Specimen, I (11)].

197 Compare Szabo [Szabo].

198 Kästner stressed that he followed the procedure Euler had given in the *Mechanica*. "Da ich aus Hrn. Eulers Buche, als es noch ganz neu war, diesen Theil der höheren Mechanik durch eigenen Fleiß erlernt (...)." [Kästner, Anfangsgründe, Vorwort]. It is very likely that Châtelet also made also use of Euler's treatise in writing the *Institutions*.

199 The analysis is mainly based on Voltaire, *Elémens* (1738), Châtelet, *Institutions* (1740), Maupertuis, *Examen* (1756), Kästner, *Mechanik* (1766), Gehler, *Physicalisches Wörterbuch* (1787).

200 Obviously, this assumption is also behind d'Alembert's interpretation of the difference between the integrals $\int K dt$ and $\int F ds$ [d'Alembert, Traité]. The argumentation is only valid for the case $K = F =$ const.

either in a pressure or a motion. Later, Euler's assumption was preserved by Kästner [Kästner, Anfangsgründe, § 52][201] whereas Châtelet maintained the Leibnizian distinction between dead and living forces [Institutions, Ch. XX and XXI].

> Man ersieht hieraus, dass jede Kraft eine doppelte Wirkung auf den Körper ausübt; die eine, wodurch sie ihnen ein gewisses Bestreben, sich zu bewegen, mittheilt und die andere wodurch diese wirklich zur Bewegung gelangen. Jene wird vorzugsweise in der Statik betrachtet und muss durch das (...) Gewicht gemessen werden. (...) Die andere Wirkung muss aber durch die Beschleunigung oder die Zunahme der Geschwindigkeit (...) gemessen werden. [Euler E015/016, § 213]

Here, we have to take into account Euler's relational definition of forces which is based on Newton's 3rd Law[202] that action always equals reaction,[203] i.e. FORCE(A, B) = FORCE(B, A), as far as the magnitude is concerned. The commonly accepted solution of the first problem had been obtained by establishing a body–force problem for *one* body and *one* impressed moving force formulated in terms of the 2nd Law, i.e. by ignoring the FORCE(A, B) = FORCE(B, A) relation. Assuming the general validity of that relation, the body–body problem has to be represented either by relative rest (or mutual pressure) or by relative motion (see above [Euler E015/016, § 213]).

Euler's contemporaries assumed the body–force relation (based on Newton's absolute time, space and motion) and simultaneously Leibniz's the relational theory of space [Châtelet, Institutions], [Kästner, Anfangsgründe].[204] Although in the debate on the foundation of mechanics by Newton and Leibniz

[201] "Die Wirkung einer Kraft, besteht entweder in einer wirklichen Bewegung, oder in einem blossen Drucke, wenn jene gehindert wird." [Kästner, Anfangsgründe] Here, Kästner established rightfully a relation between statics (or rest, i.e. relative rest) and motion, but transferred only *partially* the basic rules of *relative* rest to *relative* motion. The reason is that only the body-force relation is considered instead of the body-body relation since a pressure is mutually exerted by two bodies.

[202] Mach stressed that the 2nd Law can be derived from the action = reaction principle [Mach, Mechanik, p. 241]. The *force of inertia* and the *impressed force* are *simultaneously* present *within* the body. From the Axiom 3 it follows that the angle between these forces is zero as far as the *change in motion* is concerned, i.e. according to the 3rd Law (action = reactio) the force of inertia is acting in the same direction in which the moving force is impressed. As mentioned by Mach, the reformulated 3rd Law ("Erfahrungssatz: Gegenüberstehende Körper bestimmen unter gewissen, von der Experimentalphysik anzugebenden Umständen aneinander entgegengesetzte *Beschleunigungen* nach der Richtung ihrer Verbindungslinie") includes the 2nd Law [Mach, *Mechanik*, p. 241]. Mach added: "Das Pleonastische, Tautologische, Abundante der Newtonschen Aufstellungen wird übrigens psychologisch verständlich, wenn man sich einen Forscher vorstellt, der, von den ihm geläufigen Vorstellungen der Statik ausgehend, im Begriff ist, die Grundsätze der Dynamik aufzustellen."

[203] Leibniz assumed a soft version of the principle and claimed that "an action is always accompanied by a reaction" ("et omnis action sit cum reactio"), but stressed simultaneously the equality of the power of cause and effect ("nec plus minusve potentiae in effectu quam in causa contineatur") [Leibniz, Specimen, I (11)]. Leibniz did not stress the equality of action and reaction.

[204] Compare Châtelet's "Motion as illusion" [Châtelet, Institutions] and Kästner's "Spitzfindigkeiten" [Kästner, Anfangsgründe].

almosts all leading scientists were involved[205] the first reliable result was only obtained within the new frame established by Euler.

Euler surpassed all his contemporaries including Châtelet since he stepped beyond the intermediate stage of translation and comparison, but built from the very beginning a new foundation using the basic components of Descartes', Newton's and Leibniz's procedure in mechanics,[206] mathematics[207] and methodology.[208] The title of his comprehensive treatise on mechanics represents simultaneously his program for mechanics: *Mechanica sive motus scientia analytice exposita*. Euler introduced the following essential topics which make his mechanics different from the theories of his predecessors, (i) the rigorous statement on the priority of relative motion,[209] combined with the introduction of an *observer*, called *spectator* [Euler E015/016, § 97 and §§ 7 and 80] in the preceding paragraph of his program for mechanics [Euler E015/016, § 98], comprehensively elaborated in the *Anleitung* where the observer is called *Zuschauer* [Euler E842, §§ 77–83] and maintained in the

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[205] "Eben so wird durch die cartesianische Berechnung etwas ganz anders, als durch die leibnitzische, ausgemessen. Wenn aber doch beyde Theile das Ausgemessene Kraft nannten, so nahmen sie dieses Wort in verschiedener Bedeutung; und dieser Streit, an dem so viele scharfsinnige und gelehrte Naturforscher Theil genommen haben, war im Grunde nichts mehr, als ein bloßer Wortstreit. Das Ansehen des Herrn von Leibnitz hat diesen Behauptungen viele Anhänger und Vertheidiger verschaft, unter welche vorzüglich Daniel Bernoulli (Examen principiorum Mechanicae, in Comm. Petrop. To. I. p. 130 sqq.), Johann Bernoulli (Discours sur le mouvement, in Opp. To. III. num. 135. ingl. De vera notione virium vivarum, in Act. Erud. Lips. 1735. Maj. p. 210 und Opp. To. III. num. 145.), Hermann (Phoronomia, Amst. 1716. 4.), Bilfinger (De viribus corpori moto insitis, earumque mensura, in Comm. Petrop. To. I. p. 43 sqq.), Wolf (Principia dynamica, in Comm. Petrop. To. I. p. 217 sqq.), s'Gravesande (Physices Elem. math. L. I. c. 22. § 460.), und Musschenbroek (Introd. ad philos. natur. To. I. § 272 sq.), gehören. Dagegen ist die cartesianische Ausmessung durch MC von Mairan (Diss. sur l'estimation et la mésure des forces motrices des corps, Paris, 1741.), Iurin (Principia dynamica, Philos. Transact. no. 476 u. 479.), Desaguliers (Course of experimental philosophy, Lond. 1745. 4. Vol. I.), Maclaurin (Account of Sir Isaac Newton's philos. discoveries, Book II. Chap. 2.), Heinsius (Diss. de viribus motricibus, praeside *Hausen*, Lips. 1733. 4.) und Andern vertheidiget worden. Die Geschichte des Streits erzählen Arnold (Diss. duae de viribus vivis earumque mensura. Erlang. 1754. 4.) und noch kürzer Herr Kästner (Anfangsgr. der höh. Mech. III. Absch. § 202 u. f.)." [Gehler, Physicalisches Wörterbuch, 1787] The contribution of Châtelet had been mentioned by Kästner [Kästner, Anfangsgründe, § 203].

[206] These are the preservation of states and momentum, the relation between Newton's and Leibniz's approach, the relations between different types of forces.

[207] This concerns Euler's rediscovering and reconstruction of the unpublished parts of Leibniz's foundation of the calculus. The development of mathematics can be compared to the later rediscovering of basic principles of logics by Boole, De Morgan and Russell in the 19th century.

[208] These are the principles of contradiction and sufficient reason, the distinction between necessary and contingent truths, the logical consistency of mechanical theories, the need and the creation of special language for science.

[209] Euler rejected absolute motion which had been considered by Newton. This difference to Newton is not acknowledged, in contrary, it is often said that Euler is an adherent of absolute motion, e.g. "This is clear from the above passage, where Kant deliberately refrains from endorsing the Newtonian conception – adopted by Euler – of *absolute motion*" [Friedman]. Euler discussed and criticized Leibniz's relational theory of space and times in his paper *Réflexions sur l'espace et le tems* [Euler E149].

Theoria [Euler E289, §§ 1–11], (ii) the introduction of more than one observer[210] who are comparing the results of their observations, which results in the confirmation of (iii) the invariance of the equation of motion,[211] (iv) the explanation of the origin of forces and (v) the harmony between mathematics and mechanics resulting from Euler's procedure to coordinate his the progress in mathematics with his progress in physics. Euler transferred essential methodological rules from mathematics into mechanics and from mechanics into mathematics.

> Cette méthode d'embrasser ainsi toute les branches des mathématiques, d'avoir, pour ainsi dire, toujours présentes l'esprit toute les questions et toute les théories, était pour M. Euler une source de découverte fermée pour presque tous les autres, ouverte pour lui seul. [Condorcet, Eulogy]

Euler's summary from 1750 in the *Anleitung* has not been mentioned in contemporary treatments.[212] The *Anleitung* may be considered as an almost complete representation of all mathematical and mechanical principles which form the fundament of mechanics. The contemporary view back at this development in the 18th century was mainly formed in the 19th century and in the first half of the 20th century.

In 1756, Maupertuis summarized the development of mechanics between 1650 and 1750 [Maupertuis, Examen].

> LIX. Une aussi grand homme que tous ceux dont nous avons parlé a tiré le loix de la communication du mouvement d'un principe tout différent des précédents, & qui paroit bien plus qu'eux en être la véritable source. Mr. *Euler* a déduit ces loix du principe découvert par

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[210] Usually, the introduction of observers is not mentioned in literature. It is impossible to assume absolute motion and the assumption that all our observations are related to our positions, i.e. the position of the observers or Zuschauer, "dem Ort unseres Aufenthaltes". Moreover, Euler analysed the observations of different observers who compare their theories and their measurements (where "schätzen" is equivalent to "measure").

[211] This method has been later rediscovered and renewed by Einstein as a fundamental principle of physics.

[212] Euler analyzed the different measures in the Chap. 7 of the *Anleitung* entitled *Von der Wirkung der Kräfte auf die Geschwindigkeit der Körper* [Euler E842]. Euler formulated the problem in the spirit of Newton who considered the relations between *errors, time* and *forces* [Newton, Principia]. Following Leibniz (1686) and Newton (1687) there are two different relations, the first is a relation between mass, velocity, force and *time* (temporal interval) whereas the second one is a relation between mass, velocity, force and *space* (spatial interval). Thus, the relation between the Cartesian and the Leibnizian measure is based on a relation between time and space or temporal and spatial intervals, respectively. "Das erstere wird nämlich die *Grösse der Bewegung*, das andere die *lebendige Kraft* genannt. Ob nun gleich dergleichen Bezeichnungen willkürlich sind, so kann doch die letztere hier füglich nicht stattfinden, nachdem wir einmal für das Wort Kraft einen bestimmten Begriff festgesetzt haben. (…) Im übrigen aber, wenn man sich an den hier gegebenen Begriff einer Kraft festhält, so fallen alle Schwierigkeiten, welche sich bei dem Streite über die lebendigen Kräfte ereignen, von selbst weg und die beiden gefundenen Formeln müssen in allen Fällen die Wahrheit anzeigen." Euler's summary results from the demonstration of the *necessity* of these two representations. "Man kann auch nicht auf eine unbedingte Art sagen, wie eine grosse Kraft erfordert werde, um einen bewegten Körper in Ruhe zu bringen, indem eine jede Kraft dieses zu leisten im Stande ist; soll es aber in einer gewissen und bestimmten Zeit geschehen, so haben die Recht, welche sagen, die Kraft müsse sich verhalten wie die Grösse der Bewegung; soll es aber in einem bestimmten Wege geschehen, so haben die andern Recht, und in dieser Absicht läuft die ganze Sache gemeiniglich auf einen blossen Wortstreit hinaus." [Euler E842, § 61]

> Galilée, & reçu aujourd'hui de tous ceux qui traient de la mécanique, & de la dynamique: ce principe est que *la force multipliée par l'instant de son application donne l'incrément de la vitesse*. De ce principe Mr. *Euler* par une analyse sublime & rigoureuse tire les loix de la communication du mouvement (Comment. De la Académie de Russe, Tom IX).
>
> LX. Mais ce principe est-il une vérité nécessaire? Le seul mot de *force* qui y est employé ne l'exclut-il pas de l'ordre de ces vérités? [Maupertuis, Examen]

Although Euler demonstrated the relation between the equation of motion and the principle of least action, Maupertuis did not acknowledge the progress in the foundation of mechancis. Maupertuis' review demonstrates that Euler's theory was not a commonly accepted or comprehended basis for the communication. The representation of Euler's law by Maupertuis is incomplete. In 1756, twenty years after the publication of the *Mechanica* and some years after the paper *Découverte d'un nouveau principe de Mécanique* [Euler E181] and other papers,[213] Maupertuis noticed that Euler had oibtained by a "sublime and rigorous analysis" the principe that "the force multiplied with the increment of time gives the increment of velocity" [Maupertuis, Examen, LIX] and repeated d'Alembert's objection from 1743 on the "necessity" of the principle. Obviously, Maupertuis left out that the increment of the velocity is not only proportional to the force, but also dependent on the inverse ratio of mass.

Euler's explanation of inertia and forces was correlated with the invention of the notion of state which is described by the velocity $v = \text{const}$ including the state of rest. The states of bodies are always defined in the frame of a many body model since rest and motion are always assumed to be *relative* rest and *relative* motion. This concept of *relative* motion was completed by Newton's persisting theory of *absolute* time and space. Although Leibniz intended to demonstrate that time and space are relational notions, Newton's concept formed a reliable basis. As it had been argued by Newton (Clarke), Leibniz's theory is lacking of quantitative determination of the relations between time and space and gave only a qualitative description as orders (compare Chaps. 2 and 6). Motion turned out be independent of any constraints introduced by time-space relations and, vice versa, the time intervals and space intervals were independent of motion. As a consequence, the velocity of a moving body is not limited in magnitude, i.e. it can be defined as the ratio $v = \Delta s/\Delta t$ of arbitrarily chosen temporal intervals Δt and spatial intervals Δs. Therefore, Euler

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[213] *E145 Recherches sur les plus grands et plus petits qui se trouvent dans les actions des forces* presented 1748. *E146 Réflexions sur quelques loix générales de la nature qui s'observent dans les effets des forces quelconques* presented 1748. *E176 Expose concernant l'examen de la lettre de M. de Leibnitz alleguee par M. le Professeur Koenig, dans le mois de mars, 1751 des Actes de Leipzig, a l'occasion du principe de la moindre action* presented 1750. *E181 Recherches sur l'origine des forces* presented 1750. *E182 Lettre de M. Euler a M. Merain* presented 1750. *E186 Dissertation sur le principe de la moindre action, avec l'examen des objections de M. le Professeur Koenig faites contre ce principe*. *E197 Harmonie entre les principes généraux de repos et de mouvement de M. de Maupertuis* presented 1751. *E198 Sur le principe de la moindre action* presented 1751. *E199 Examen de la dissertation de M. le Professeur Koenig, insérée dans les actes de Leipzig, pour le mois de mars 1751* written 1752, presented 1751. *E200 Essai d'une démonstration métaphysique du principe général de l'équilibre* written 1753, presented 1751. [The Euler Archive]

related the equality of temporal and spatial intervals as well as the preservation of direction to the space [Euler E149, §§ 20–21].[214]

The same correlation between the theoretical model and experience appeared in case of the notion of inertia and the notion of forces where the *relational* and the *non-relational* model had been also discussed simultaneously. Newton and Leibniz invented two kinds of forces, first the *inherent* forces like (i) the force of inertia (Newton), (ii) primitive forces (Leibniz) and, second forces of *relational* or *quasi-relational* type like (iii) the derivative (Leibniz) and (iv) the impressed moving forces (Newton). Additionally and by metaphysical reasons, Leibniz assumed (v) active and passive and (vi) dead and living forces (compare Chap. 2). However, the classification by inherent and relational forces had not been made in advance by postulating a methodological principle, but more or less implicitly referring to the classification by the distinction between substances and accidents [Euler E842, § 18].[215] Following Leibniz, the contemporaries of Euler's comprehended *inertia, forces* and *motion* as properties of bodies. Furthermore, although the properties of bodies are experimentally explained by the actions of an observer, the equivalence of the observer and any other body is not treated as an essential part of the theory. Kästner explained the inertia by a body–observer interaction ("so fühlen wir")[216] whereas Euler made exclusively use of a body–body interaction.

> 22. Wenn wir einen Körper der still zu liegen scheinet nur auf einer wagrechten Ebene fortschieben wollen, so fühlen wir daß eine Gewalt dazu nöthig ist. 23. Heißt man also Zustand eines Körpers, die Ruhe wenn er sich nicht bewegt, Richtung und Geschwindigkeit, wenn er sich bewegt, so finden wir, daß wir ohne Anwendung einiger Gewalt, den Zustand

[214] "20. The situation is the same with respect to the equality of time; for if time is nothing but the order of successions, how can one make the equality of time intelligible?
21. The question here is not our estimation of the equality of time, which without doubt depends on the state of our mind; but the question is the equality of time during which a body put in a uniform movement runs equal spaces. Since that equality cannot be explicated in terms of the order of successions, no more than the equality of space in terms of the order of coexistents, and that equality is essentially involved in the principle of movement; one cannot say that the bodies, in continuing their movement, depend on a thing which does not exist but in our imagination." [Euler E149 (Uchii)]

[215] "18. Insofern also durch die Bewegung bloss allein das Verhältniss, in welchem der äussere Umfang eines Körpers mit dem Raume steht, verändert wird, so leidet daher der innere Zustand eines Körpers keine Veränderung und demnach kann die Bewegung weder unter die Eigenschaften noch Zufälligkeiten eines Körpers gerechnet werden." [Euler E842, § 18]

[216] Kästner did not describe his action upon the body as a *relative motion* between himself and the body. The complete description of the setup, formed by a body and an observer who acts upon the body and the body simultaneously acts upon the observer, and the observed effects had been already completely given by Leibniz in 1695 (unpublished version of *Specimen*). The crucial point is the complete description of relative motion.
"Sequitur etiam ex natura motus respectiva *eandem esse corporum actionem in se invicem seu percussionem, modo eadem celeritate sibi appropinquent*, id est manente eadem apparentia in phaenomenis datis, quaecunque demum sit vera hypothesis seu cuicunque demum vere ascribamus motum aut quietem, eundem prodire eventum in phaenomenis quaesitis seu resultantibus, etiam respectu actionis corporum inter se. Atque hoc est quod experimur, eundem nos dolorem sensuros sive in lapidem quiescentem es filo si placet suspensum incurrat manus nostra, sive eadem celeritate in manus quiescentem incurrat lapis." [Leibniz, Specimen, II (2)]

> eines Körpers nicht ändern können, und nehmen also dieses für das Merkmahl an daran wir Körper erkennen. [Kastner, Anfangsgrunde, §§ 22 and 23]

Motion is interpreted as a (body–body) – (non-acting observer) relation, force is interpreted as (body) – (acting observer) relation. Euler defined forces in a frame composed of (i) (non-acting body–non acting body) – (non-acting observer) relation which is completed by a (ii) (acting body–acting body) – (non-acting observer) relations. The Eulerian observers are never acting upon the bodies and the bodies are never acting upon the observers whereas the bodies are either not acting or acting upon each other [Euler E842, § 49]. Following Euler, Kästner discussed other external causes for the change of the state [Kästner, Anfangsgründe, I (26)] and replaced the body-acting observer model by the body-acting external cause model (the body is only reacting until now). The inertia is never solely derived from the preservation of states and the body–body interaction where the latter one modifies the states of both bodies. Nevertheless, Kästner's analysis is as interesting as Châtelets representation of the different versions of basic notions of mechanics. Both authors intended to join the ideas of the different schools. Euler's inventions of new concepts in the *Mechanica* and *Theoria* is acknowledged, but only incompletely recognized. As a consequence, the reader can convince himself that these ideas of different schools are not compatible to each other and, moreover, not compatible to Euler's approach.

> 23. Es sieht also wenigstens so aus, als ob in dem Körper etwas wäre, das ihn in dem Zustande in dem er ist zu erhalten strebte und jeder Bemühung ihn aus diesem Zustand zu bringen hinderlich fiele, und den Zustand nicht eher bis es überwunden ist,[217] ändern ließe. Dieses kann man die Trägheit (inertiam, vim inertiae)[218] nennen. (...) Diese sogenannte Kraft, hat vieles, das selbst die für wunderbar und von der Vorstellung anderer Kräfte ganz abweichend erklären, die ihr am meisten gewogen sind. Sie übt niemals für sich eine Würkung aus, nur zeigt sie wenn sie gleichsam aufgefordert wird,[219] eine Gegenwirkung. (...) So ist die Kraft, so zu reden nur ein Wiederhall anderer Kräfte.[220] (...) Sie hat auch keine bestimmte Größe denn sie wiedersteht viel oder wenig, nachdem die Gewalt der sie sich wiedersetzet groß oder klein[221] ist [I (21)]. (...) So ist die Trägheit weiter nichts als der Satz des zureichenden Grundes auf die Veränderung des Zustandes der Körper angewandt.[222] Wer behauptet, dass sie aus dem Begriff des Körper fliesse, der hat völlig

[217] The inertia is a temporarily existing property.

[218] Following Euler, Kästner commented on the name "force of inertia", but interpreted the "inertia" differently.

[219] Here, Kästner adopted he interpretation of Newton [Newton, Principia, Definitions] and Leibniz [Leibniz, Specimen, II (5)]. Obviously, the problem is caused by the interpretation of the inertia as a "force".

[220] Once more, this is the Newton-Leibniz interpretation mentioned before. It becomes clear that Euler stepped beyond Newton and Leibniz by the introduction of a new concept of forces and the correlated concept of mass (compare Jammer [Jammer, Mass]).

[221] Therfore, the magnitude of inertia is not related to the magnitude mass of the body, i.e. it is not an invariant property of the body, but its magnitude depends, according to the pre-Principia model assumed by Newton [Westfall, Newton, p. 220], on the magnitude of the external power or puissance which is impressed upon the body. Kästner assumed that the *inertia* of a body is not an *internal* and *invariant* property, but depends on the magnitude of "force impressed" upon the body.

[222] Compare the same procedures applied by Châtelet who made also use of the principle of sufficient reason [Châtelet, Institutions].

> Recht denn wir haben uns diesen Begriff nach unseren Empfindungen gemacht, und folglich die Trägheit mit hinein gebracht.[223] [Kästner, Anfangsgründe]

From the analysis of Kästner's *Anfangsgründe* we obtain an interesting insight in the thinking of the scholars on basic concepts of mechanics between 1730 and 1760. The result can be summarized as follows[224]: (i) In the first half of the 18th century, the theories of Newton and Leibniz are simultaneously reconsidered and recognized, (ii) after 1740, Euler's *Mechanica* is used as a summary of the state of art and as a basis of the further development, (iii) Newton, Leibniz and Euler are only partially acknowledged and accepted,[225] (iv) new concepts had been introduced by d'Alembert and Maupertuis, (v) Euler created a new consistent and almost complete representation of mechanics[226] based on a unification and merging of Newton's and Leibniz's legacy. Although Kästner studied carefully Euler's treatise, he could not reproduce the full content of Euler's *Mechanica* especially the innovations as far as the relations between inertia and forces are concerned, but it should be mentioned that Euler also explicitly elaborated in more detail the hidden subtext of *Mechanica* only some years later in the *Anleitung* [Euler E842]. However, the difference between the papers resulting from the reception of Euler's ideas by the contemporaries and Euler's own later contributions demonstrates the difficulties and obstacles to overcome the resistance of the persisting incompatible schemes presented by the different schools.

4.4 Euler's World Models

Euler modified the Newtonian axiomatics by a correlated and distinctive treatment of rest and motion and completed the Leibnizian axiomatics by transmutating the Leibnizian metaphysical construction of the world into a tool of analytical and rational mechanics:

(1) the conservation of state is a generalization of the Leibnizian assumption on the equivalence of "cause and effect";
(2) the conservation of state is not traced back to any metaphysical law or to the creation of the world by God.

The subject of the theory is preferentially the explanation of the *change of state*. The conservation of state is solely explained by the absence of any external cause being able to change the state [Euler E842, § 1]. The other typical feature of Euler's theoretical approach can be demonstrated by the comparison with Newton's

[223] Later between 1770 and 1781, Kant generalized the procedure described by Kästner and assumed that "we are giving the laws to nature, i.e. to bodies" which suffer from the lack of any own internal laws. If there are such laws we can never get knowledge of them.

[224] The same conclusion is obtained from the analysis of Châtelet's *Institutions* [Châtelet, Institutions].

[225] Compare d'Alembert's comment on Euler's *Mechanica* [d'Alembert, Traité]

[226] Compare Preface to *Mechanica* [Euler E015/016].

Principia. Euler reduced the Newtonian set of axioms and derived both its 2nd and its 3rd laws instead of assuming them to be axioms. This feature has been underestimated or even ignored during the later development of mechanics. Furthermore, Euler completed the Newtonian approach by the introduction of the so called "indirect method" for solving mechanical problems, which is closely related to Maupertuis' principle of least action. The indirect method can be interpreted as a problem of choice of a certain path of a moving body obtained by minimizing the forces. Therefore, the methodology introduced by Leibniz for the explanation of God's choice of a certain "world" has been transmuted by Euler into a tool of rational mechanics.

The concept of the multitude of "possible worlds" has been introduced by Leibniz for the explanation of God's decision to create a real world after his preceding selection of this special world from the infinitude of possible worlds. The construction principle for this multitude is explained in the *Theodicy* named by Leibniz "my principle of an infinity of possible worlds" [Leibniz, Theodizee]. The crucial point is the assumption that any "world" is to be considered as a model for special physical laws, so that in different worlds, different types of individuals as well as different types of laws may exist.

Leibniz did not construct such different types of mechanical worlds in detail, since he was only interested in the demonstration of the general principles for the choice and the creation of a special world by God. Nevertheless, the methodological potential of Leibniz's concept is not exhausted by this demonstration of a general reason for God's choice. As mentioned above, Euler developed the Leibnizian methodology just in that sense to construct different types of mechanical "worlds" or "models". He used these models for the demonstration of basic principles of mechanics by thought experiments. We will review shortly the most important models which played an essential role in the controversy with the Wolffian school [Euler E081].

Following Descartes, Euler distinguished between spirits (souls) and bodies whose properties exclude each other. Therefore, the basic laws for bodies and souls can be analyzed independently of each other and the world of bodies forms a realm of its own [Euler E842, § 49], [Euler E344, Lettre LXXXV]. Beside rest and motion, the invariant property of bodies is the impenetrability [Euler E842, Chaps. 2, 3, 4 and 5]. Although "impenetrability is not capable of a certain magnitude" whose numerical value is known, it is of the *same* magnitude for all bodies albeit this magnitude is *indeterminate*. Euler introduced a mechanical notion which cannot be directly, but only indirectly related to experiment. Supposing the generation of forces by the bodies, it is unavoidable to assign the impenetrabilty (or, as Newton did, the universal attraction) to the bodies as a general property which is independent of the shape and the composition of bodies. The impenetrability defines a limit for the strenght of interaction which is removed if the bodies penetrate each other. Euler assumed that it is impossible to destroy the bodies by overcoming the impenetrability. Hence, Euler introduced an upper limit for those forces which are generated by bodies. The bodies are not able to destroy themselves nor remove their impenetrability by natural means. Moreover, also God cannot remove the impenetrability from bodies [Euler E344, Lettre LXXXV].[227]

Euler based his theory on the complete Cartesian frame made up of three basic notions, i.e. the distinction between (i) res infinita, (ii) res cogitans and (iii) res extensa, and interpreted the relation between (ii) and (iii) as being defined by a logical opposition. Although ghosts and bodies are of completely different nature, their relations can be adequately expressed in terms of statements which are connected of different type, e.g. (i) "A and B", (ii) "A or B", (iii) "either A or B", (iv) "neither A nor B" &c.

> But spirits are of a very different nature, and their actions depend on principles directly opposite. [Euler, E344, Lettre LXXXV]

The *direct opposition* is expressed by the statement that all actions are "either those of bodies or those of ghosts" or, following Schrödinger by "the full force of logical opposition".[228] Hence, assuming that the world is made up of spirits and bodies, each of things in the world is "either a spirit (soul) or a body".[229] The result is a necessary relation between spirits and bodies whose validity ensures their stability and the distinction from each other. The definition of a body (by extension or impenetrability) would be *incomplete* and questionable without the simultaneous definition of a ghost (by the complete absence of extension or impenetrability) and vice versa, the definition of a ghost would be incomplete in case of the lack of bodies. Hence, correlating the *quantified* inertia (by the inert mass of the bodies) and the impenetrability being "not capable of quantification", Euler assumed indirectly that a quantification of the impenetrability should be due to the application of methods being different from those which are appropriate for the quantification of inertia (mass).

Euler constructed different models being as simple as possible to demonstrate the basic laws of mechanics.

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[227] "But spirits are of a very different nature, and their actions depend on principles directly opposite. Liberty, entirely exclude from the nature of body, is an essential portion of spirit, to such a degree, that without liberty a spirit could not exist; and this it is which renders it responsibility for its actions. This property is as essential to spirits as extension or impenetrability is to body; and as it would be impossible for the divine Omnipotence itself to divest body of these qualities, it would be equally impossible for it to divest spirits of liberty. A spirit without liberty would no longer be a spirit, as a body without extension would no longer by a body." [Euler E344, Lettre LXXXV]

[228] In 1933, Schrödinger analyzed the paths for classical and quantum particles and formulated the result in terms of a logical statement. "We are faced here with the full force of the logical opposition between an either – or (point mechanics) and both – and (wave mechanics)" and concluded: "This would not matter much, if the old system were to be dropped entirely and to be *replaced* by the new. Unfortunately, this is not the case." [Schrödinger 1933] (compare Chap. 8)

[229] A body is either resting or moving. "A thing which is neither resting nor moving is not a body". "3. Nullum enim existere potest corpus, quod non vel moveatur vel quiescat." [Euler E015/016, § 3]

Model 1: The simplest case is a world, where only one body exists [Euler E181, § 2] (1750, 1752), [Euler E343, Lettre LXXI][230] (1760–1761). The world is composed of one body and the empty space being infinitely extended.

> 2. En effet si nous ne considérons qu'un seul corps, en supposant que tout le reste du monde soit anéanti, et que ce corps existe tout seul dans l'espace vuide et infini, la vérité de ce que je viens d'avancer sur la conservation de l'etat, sautera d'abord aux yeux. [Euler E181, § 2][231]

One decade later in 1760, Euler referred to this model to demonstrate the basic principles in the theory of motion.

> That hypothesis, in spite of being impossible, allows us to distinguish between the properties of the body itself and that, what will happen in case of other bodies operate on it. [Euler E343, Lettre LXXI][232]

It is assumed that the body is either at rest, or in motion, i.e., that the body is always in a certain state. This state cannot be changed by the body itself. The cause of a change of state is located outside the body.

In Letter LXXI, important methodological principles are reviewed and simultaneously applied to demonstrate the reliability of the mechanical theory. Euler claimed that (i) hypotheses are tools of investigation, (ii) the philosophers and, consequently, mathematicians and physicist, are obliged to create a particular language different from the commonly used language,[233] (iii) the intention is to give a foundation of the theorems by methods being equivalent to the demonstrations of geometric truths[234] and, finally, (iv) the equivalence of the states of rest and motion and their

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[230] Equivalence between rest and motion is demonstrated by (i) the equivalence of the syntactic and logic structure of the basic statements and (ii) by the same analytical form of the equation of motion (compare Chap. 6). The change of velocity is independent of velocity. Usually, there is a difference between the *syntactic* structure and the *logical* structure of statements and statements composed of different (elementary) statement.

[231] Euler referred to Archimedes' method to study the laws of the lever at first in empty space and at second in the real world. It is assumed that the relations between the bodies in empty space are preserved, but the phenomena are modified by additional influences [Euler E015/016, § 56]. In the 17th century, Galileo developed the same method to studying falling bodies of different shape and masses in vacuum and in air.

[232] "Cette hypothese, quoique impossible, peut faire distinguer ce qui est opéré par la nature du corps même, de ce que d'autres corps peuvent opérer sur lui." [Euler E343, Letter LXXI] The essential role of hypotheses had been also stressed in the letter to Châtelet. "Mais surtout le Chapitre sur les hypotheses m'a fait le plus grand plaisir, voyant, que Vous combattez, Madame, si fortement et si solidement quelques Philosophes Anglois, qui ont voulu banner tout à fait les hypothèses de la Physique qui sont pourtant à mon avis le seul moyen de parvenir à une connoissance certaine des causes physiques." [Euler, Correspondence with scholars, Lettre to Châtelet]

[233] "Toutes les langues sont introduites pour l'usage du people, et les Philosophes sont obligés de se former une langue particuliere." [Euler E343, Lettre LXXI]

[234] Euler commented on the conservation of state, either the state of rest or the state of uniform motion in a straight direction. "Quelque fondée que soit cette loi, qui pourroit aller de pair avec les vérités géométriques, il y a des gens peu accoutumés à examiner les choses, qui prétendent que l'expérience y est contraire." [Euler E343, Lettre LXXI] *Force* is defined as an external cause, i.e. all kinds of *inherent* forces are finally excluded [Euler E343, Lettre LXXIV].

changes by external causes is not only expressed mechanically and mathematically, but also represented by the syntax and semantics of the statements on the preservation of states.[235] Hence, Euler anticipated essential methodological principles which had been later developed by analytical philosophy.

The bodies change their states *mutually*, without any reference to the world as a whole. This type of treatment is a real novelty, when compared with Leibniz' methodology to consider the individuals only with respect to the world as a whole. Moreover, Leibniz hardly formulated any conservation rule with respect to the single bodies (since he had excluded very early the inertia from the set of basic concepts). Moreover, a simplest world does not really exist in the hierarchy given by the pyramidal picture of the infinitude of possible worlds. Leibniz claimed, that the pyramid has a tip, but no basis [Leibniz, Theodizee, § 416], i.e. there is no such thing as the *simplest* possible world. Nevertheless, Euler made use of the Leibnizian model, but interpreted it quite differently.

Following Euler, the world consisting only of one body and the empty space may be regarded as a model for the simplest world. Adding further bodies, each of these additional parts have to have the same properties as the body from the simplest model since all other bodies can be removed and any of the bodies will form together with the empty space an equivalent world model. Any decisive change can be only introduced by the addition of a second body. Then, except the *body–vacuum* relation (which is not modified) additionally a *body–body* relation has to be defined. Assuming that a place cannot be occupied simultaneously by more than one body, the second body has to be positioned at a place being *different* from the place the first body is occupying. Furthermore, as a consequence, the first body cannot occupy the place of the second body.

The basic assumptions on the body and the empty space are not modified if additional bodies are added to the "world" although other worlds being different from them simplest one are obtained.

Using the Archimedean procedure to study the equilibrium of the lever in empty space (Fig. 4.3) and to transfer the setup into the real world, Euler demonstrated the basic principles of mechanics in relation to different world models.

Model 2 is a world, where only two bodies exist [Euler E343, Lettre LXXI]. Since the state of a body can be changed only by an *external cause*, the external reason is provided by the other body. The interaction takes place, because the bodies *occupy* a certain space region and, moreover, a given space region *can only be occupied by one body*. It is impossible, that two bodies occupy the same space region (Euler's exclusion principle). If two bodies interact, the *states of both bodies are changed mutually* (Fig. 4.4).

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[235] "Or on énonce communément ce principe pat deux propositions, dont l'une porte, *qu'un corps étant une fois en repos demeure éternellement en repos, à moins qu'il ne soit mis en mouvement par quelque cause externe ou étrangere*. L'autre proposition porte *qu'un corps étant une fois en mouvement, conservera toujours éternellement ce mouvement avec la même direction et la même vitesse, ou bien sera porté d'un mouvement uniforme suivant une ligne droite, à moins qu'il ne soit troublé par quelque cause externe ou étrangere*.

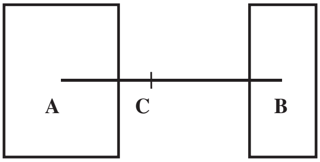

Fig. 4.3 Archimedes on the balance between two weights
Archimedes (Heath), Proposition 3 Book I *On the Equilibrium of Planes*

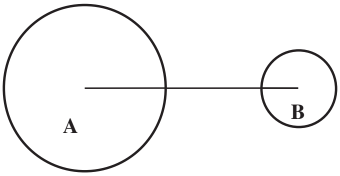

Fig. 4.4 Euler on the interaction of two bodies. The world is made up of two bodies [Euler E343, Lettre LXXVII]
(...) il suffira de considérer deux corps, comme s'ils existoient seuls au monde.

Euler demonstrated all basic principles of his mechanics within the framework of these two models. All other, more sophisticated constructions do not need any additional basic principles.
Model 3: All bodies in the world are at rest [Euler E081, II, § 13].
Model 4: All bodies in the world are moving with equal velocity in the same direction [Euler E081, II, § 12]. They preserve the *same relations* and *the same order* among themselves. Hence, no change will appear within this world. Thus, in this case, the conservation of the state is a fundamental property of bodies, again.

These world models[236] are correlated to the *analytical* representation of the motion of bodies by a system of differential equations, first a system of correlated, but uncoupled and, second a system of correlated and coupled differential equations of first order.
World A consisting of two non-interacting bodies B1 and B2.

(a) $m_1 dv_1 = 0$ and $m_2 dv_2 = 0$,
(b) hence $v_1 = \text{const}$ and $v_2 = \text{const}$,
(c) with $v_1 = v_{12}^{\text{relat}} = \text{const}$ and $v_2 = v_{21}^{\text{relat}} = \text{const}$,
(d) $|v_{12}^{\text{relat}}| = |v_{21}^{\text{relat}}| = \text{const}$.

The items (a) and (b) are mechanically and mathematically correlated by Newton's 1st axiom and the representation in terms of the Newton-Leibnizian differential calculus in Euler's interpretation.[237] Item (c) follows from Euler's interpretation

C'est en ces deux propositions que consiste le fondement de toute la science du Mouvement, qu'on nomme la Mécanique." [Euler E343, Lettre LXXIII]

[236] In contemporary sccience, the basic world model or the "background" is the 4D Minkowski space and the different "worlds" are constructed as distinguished representations of the Einstein-Hilbert functional [Smolin].

[237] The relations (a) and (b) follow from Euler's "General and Fundamental Principle of Mechanics" ("Principe Général et Fondamental de toute la Mécanique") [Euler E177].

of relative motion being different from Descartes' and Leibniz's theory of relative translation. In the absence of forces, the "coupling between the bodies" or the "parts of the system" is due the relation (d) since the equality of the relative velocities in magnitude is a *necessary* condition that the bodies B1 and B2 belong to the "same world". As a consequence, having only one body, it is impossible to distinguish between "rest" and "motion" since neither rest nor motion can be *assigned* to the body. The state of the body is mechanically indeterminate or inassignable.[238] In case of a multi body system the indeterminacy is only partially removed since relative velocities of different magnitude are to be assigned to each of the bodies of the system, e.g. in case of a three body system there are two different relative velocities assigned to B1. It follows $|v_{12}^{\text{relat}}| = |v_{21}^{\text{relat}}|$ and $|v_{13}^{\text{relat}}| = |v_{31}^{\text{relat}}|$ where, in general, the relative velocities are different from each other $|v_{12}^{\text{relat}}| \neq |v_{13}^{\text{relat}}|$. Hence, the bodies form a world if the "coupling relations" can be established.

World A consisting of two interacting bodies B1 and B2

(a) $m_1 dv_1 \neq 0$ and $m_2 dv_2 \neq 0$, $\quad m_1 dv_1 = K_{12} dt_1$ and $m_2 dv_2 = K_{21} dt_2$
(b) hence $v_1 \neq \text{const}$ and $v_2 \neq \text{const}$,
(c) with $v_1 = v_{12}^{\text{relat}} \neq \text{const}$ and $v_2 = v_{21}^{\text{relat}} \neq \text{const}$,
(d) $|v_{12}^{\text{relat}}|_{\text{before}} = |v_{21}^{\text{relat}}|_{\text{before}}$ and $|v_{12}^{\text{relat}}|_{\text{after}} = |v_{21}^{\text{relat}}|_{\text{after}}$

In the presence of forces, the "coupling between the bodies" or the "parts of the system" is due the relation (a). The forces are generated by the bodies due to the mutual change of their states. Hence, the change of the state is simultaneously, $dt_1 = dt_2 = dt$. Therefore, simultaneity is defined by the interaction of bodies. Moreover, the bodies are not moving relatively to each while interacting, but form a whole which decays into parts after the interaction is terminated ("coeunt" [Euler E289, § 131]).

As a result, Euler obtained a *unified concept* of bodies and forces. There is only one type of bodies and *one type of forces*. The forces do not act at distance, but appear only due to the competition for space occupation. The *conservation of state is not due to a force, but due the nature of the bodies*. Later, Kant tried to unify the Leibnizian and the Eulerean concepts of forces and bodies.[239] Then, we can concluded that Euler has based the theory of motion on the concepts of mechanics introduced by Newton *and* by Leibniz. The basic concept is the conservation of state. Leibniz used the conservation law in a metaphysical and a mechanical sense:

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[238] Usually, the inassignability had been discussed for *differentials*, i.e. mathemically defined quantities being different from finite quantities whose properties are assumed to be subjected to the Archimedean principle (compare Chap. 3 [Leibniz, Historia], [Arthur, Syncategorematic]). Now, the Archimedean principle is in power, but there are finite quantities like velocity which turn out be indeterminate. This point had been accentuated by Newton in criticizing the Cartesian relativity of motion which results in an indeterminacy of the velocity assigned to a body (compare [Westfall, Newton]).

[239] I. Kant, *Metaphysische Anfangsgründe der Naturwissenschaft*, hrsg. von K. Pollok, Hamburg 1997.

(1) as a general metaphysical principle stating the equivalence of cause and effect, where the term "metaphysical" has the meaning of being beyond geometry and not to be based on geometry only, and,
(2) as the conservation of the sum of living forces and the conservation of total motion and total direction of motion of a system of bodies. Euler used the conservation of state in a purely mechanical sense, without any additional reference to a metaphysical foundation [Euler E065, Additamentum II]

Euler derived general relations for subdividing motions into possible motions and impossible motions or changes. The so-called equation of continuity defines the conditions for possible and impossible motions [Euler E842, § 156] which had been later called *equation of continuity*. Further principles are: "It is impossible that, at the same time, one and the same body can occupy more than one place" [Euler E842, § 35]. It is impossible that a body can change the state of another body without changing its own state. These statements by Euler are made in the goal and spirit of Leibniz' construction of an infinity of possible worlds. However, Euler introduced the consideration of a *finite* instead of the *infinite* set of models of the world for the purpose of a self-consistent foundation of mechanics. The concept of infinity is treated *mathematically* and completed by the analysis of infinitely small quantities [Euler E212].

It had been demonstrated that Euler generalized the models and theories of his predecessors Descartes, Newton and Leibniz. In the following chapters it will be analyzed how Euler's model can be subjected to the corresponding procedures to make them to be the origin of further generalization.

Chapter 5
The Foundation of the Calculus

Perspicuum est Calculum differentialem, ejus, quem ante exposui, calculi esse casum specialem: nam, quod ibi quantumvis erat assumptum, hic ponetur infinite parvum.
Euler 1727, p. 164[1]

From the very beginning of his scientific career in 1727 [Euler 1727], the foundation and application of the calculus concerns an essential part of Euler's work for several decades. Euler published the complete theory in 1748 [Euler E212] and finished the series of book on the differential and integral calculus [Euler E342], [Euler E366], [Euler E385][2] finally finished in 1770. All his great textbooks on mathematics and mechanics are directly or indirectly patterned upon this matter [Euler 1727], [Euler E015/016], [Euler E065], [Euler E101], [Euler E102], [Euler E212], [Euler E814] (according to Eneström, this is an application of differential calculus to geometry, published 1862), [Euler E289], [Euler E342], [Euler E366], [Euler E385]. Beside the relationship between these disciplines, Euler accentuated the autonomy of the principles mathematics and mechanics are governed. As a consequence, mathematics and mechanics are to be built on their own fundaments which are represented by those principles of mechanics which cannot be reduced to geometry and by those principles of mathematics which cannot be reduced to mechanics. This understanding of mathematics and mechanics follows from an essential modification of the *fundamental role geometry* played as

[1] "It is evident that the differential calculus is a special case of the calculus which I have enunciated above, since what was before assumed arbitrary is now taken to be infinitely small." [Euler 1727]

[2] The reliability of the calculus was not only decisive as a foundation of a mathematical discipline, but also to same extend crucial for the foundation and analytical representation of mechanics. As in mathematics, following their predecessors Descartes, Newton and Leibniz, Euler and all other scholars had accepted the challenge to create a *theory of motion* [Euler E015/016] which turned out to be as reliable and consistent as the *theory of the equilibrium* known from the ancients. Following Newton and Leibniz (principle of continuity), the mathematical rigor had to transferred into mechanics. Hence, the *criteria* for mathematical rigor had to be defined and fulfilled at first in mathematics. Following the analysis of Felix Klein, this development in the foundation of the calculus had been initiated by Euler and finished by Weierstraß [Klein, Arithmetization] (compare Chap. 3). For the state of art in the end of the 19th century compare DIFFERENTIALRECHNUNG in *Meyers Conversations-Lexikon* 4th edition (calculus of differences and differential calculus, 1885–1892) and 6th edition (differential calculus, calculus of differences had been omitted, 1913) [Meyer].

D. Suisky, *Euler as Physicist*, DOI 10.1007/978-3-540-74865-6_5,

paradigm for *reliability* and *rigor* not only in the theories of Newton and Leibniz, but also ever since in the foundation of static, the science of equilibrium, by the Ancients[3] (compare [Leibniz, Nova methodus]). Euler paved the way for a new development of mathematics and mechanics which had been later called "arithmetization of mathematics" by Felix Klein [Klein, Arithmetization] correlated with the consequent development and application of the concept of function [Klein, Elementarmathematik].[4] The geometric rigor is not only completed by arithmetic rigor, but, moreover, replaced with arithmetic rigor. The fundamental change in the paradigmatic role of geometry becomes apparent and immediately detectable by looking at the textbook pages. Comparing Newton's *Principia* and Lagrange's *Mécanique analytique*, figures are not only replaced with calculations by the authors, but are intentionally banished from the textbooks [Euler E212, Preface], [Lagrange. Mécanique]. Following Lagrange, the demonstrations of theorems in mathematics and mechanics are based on the idea of function including the foundation of the calculus [Lagrange, Fonction],[5] [Lagrange, Works 10], [Lacroix].[6]

Euler tested this concept for mechanics giving consequently analytical demonstrations [Euler E015/016] (compare Chap. 4), but preserved these principles also for mathematics and the analysis of motion by the method of maxima and minima [Euler E065]. Euler put together all elements being necessary and sufficient for the introduction and definition of a complete set of algebraic and numerical algorithms which is independent of the interpretation and confirmation in a geometrical model.[7] In 1715, Taylor did the same essential step towards an arithmetical

[3] Leibniz simultaneously questioned and preserved the paradigmatic role of geometry by the invention of the "calculus of differences and sum" [Leibniz, Elementa calculi] and the transfer of the "principle of continuity" from geometry into mechanics [Leibniz, Specimen II (4)] (compare Chaps. 2 and 3), respectively. Euler generalized the validity of the Leibnizian principle by the inclusion of discrete quantities including finite, infinitesimal and infinite quantities represented by numbers. "98. Quantitates tum infinite parvae, quam infinite magna in seriebus numerorum saepissime occurunt, in quibus cum sint numeris finitis permixtae, ex iis luculenter patebit, quemadmodum secundum leges continuitatis a quantitatibus finitis ad infinite magnas atque infinite parvas transitio fiat. Consideremus primum serium numerorum naturalium, quae simul retro continuata erit." [Euler E212, § 98]

[4] "Wir wollen nur, daß der allgemeine Funktionsbegriff in der einen oder anderen Eulerschen Auffassung den ganzen mathematischen Unterricht der höheren Schulen wie ein Ferment durchdringe." [Klein, Elementarmathematik, p. 221]

[5] "On a transporté dans l'analyse les principes qui résultaient de ces considération, (...) envisagés analytiquement, se réduisent simplement à la recherche des fonctions dérivées qui forment les premiers termes du développement des fonctions données, ou à la recherche inverse des fonctions primitives les fonctions dérivées." [Lagrange, Fonctions]

[6] "(...) mais sans avoir aucun égard aux valeurs de ces accroissemens. Dans le calcul des différences au contraire, le but est de déterminer les accroissemens eux-mêmes, en les déduisant non seulement de l'expression analytique des fonctions, ais aussi de leurs valeurs numériques ou particulière, lorsque l'expression analytique manque." [Lacroix, § 340]

[7] The application to geometry is only in [Euler E814]. "Chapter 1: De calculo differentiali ad lineas curvas applicato in genere, Chap. 2: De tangentibus linearum curvarum, Chap. 3: De tangentibus linearum curvarum, quae per alias lineas curvas utcunque determinantur, Chap. 4: De tangentibus curvarum, in certis locis inveniendis."

representation and foundation of the calculus in the treatise *Methodus Incrementorum Directa & Inversa*[8] [Taylor, Methodus].[9]

The *algebraic* representation of infinitesimal quantities had already been invented by Newton between 1665 and 1666 [Newton, Method of Fluxions], "De methodis serierum et fluxionum" (unpublished, 1671) and Leibniz between 1675 and 1678 [Leibniz, Historia] (published only in the 19th century) and finally published in 1684 [Leibniz, Nova methodus]. There is, however, an essential difference between the representation of the new method between Newton and Leibniz as far the implicit use of the notion of function is concerned.[10] Assuming as many *differentials* as *independent* variables, Leibniz made an essential progress in comparison to Newton who restricted the set of independent variables to one variable due to the invention of only *one* universal mathematical time and, consequently, "only one infinitely little quantity *o*" [Newton, Quadrature (Harris)], [Newton (Collins), Commercium] (compare Chaps. 2 and 3).[11] Hence, Leibniz paved the way for a *purely mathematical* foundation of the calculus which is independent of the interpretation of any of the variables in terms of mechanics or geometry. Instead of the Newtonian universal independent variable "time",[12] Leibniz constructed an algorithm which is applicable to any geometrical or mechanical problem since neither the *mechanically determinate dimensions* nor the *number* of the variables are modified by the application of the "new method" [Leibniz, Nova methodus] (compare Chap. 3).

[8] "In the first part are explained the principles of the new incremental method, and by the means of that the method of fluxions is more fully explained than has yet been done; it being shown how this method is deduced from the former, by taking the first and last ratios of the nascent and evanescent increments. In the second part the usefulness of these two methods is set forth by several examples viz." This is the announcement written by Taylor [Taylor, Methodus]. [http://www-history.mcs.st-andrews.ac.uk/Extras/Taylor_continental.html]

[9] The response to Taylor's treatise was quite different. Johann Bernoulli wrote to Leibniz that there is nothing new and most of solutions are stolen [Johann Bernoulli, Response to Taylor]. Some decades later, Lagrange was in favour of Taylor's approach [Lagrange, Fonctions]. In the 19th century, also Klein acknowledged Taylor's contribution to the foundation of the calculus [Klein, Arithmetization].

[10] For the historical development of the concept of function in the 17th and 18th centuries compare Thiele [Thiele 2007].

[11] Newton paved the way for a "mechanization" of the calculus, i.e. the application to mechanics. Kepler and Cavalieri treated geometrical problems ro resolve the geometrically defined continuous extended things into indivisible elements. "Parallel mit der (...) Entwicklungsreihe, auf der sich die heutige wissenschaftliche Mathematik aufbaut, hat sich eine wesentlich verschiedene Auffassung der Infinitesimalrechnung durch die Jahrhunderte fortgepflanzt. Sie geht zurück: 1. auf *alte metaphysische Spekulationen über den Aufbau des Kontinuums* aus nicht mehr weiter zerlegbaren, 'unendlich kleinen' Bestandteilen. Als (...) Beleg nenne ich den Titel des (...) Buches *Cavalieris Geometria indivisibilibus continuorum promota*, der seine wahre Grundauffassung andeutet." [Klein, Elementarmathematik, p. 231].

[12] By this construction, Newton excluded the "discrete things" and ensured the *autonomy* of the continuum and guaranteed the axiomatic status of motion in mechanics whereas Leibniz's methodological guiding principle is to clarify the relation between the "continuum and the discrete things" [Leibniz, Theodizée].

5.1 The Arithmetization of the Calculus

The basic relations of the calculus had been almost completely invented by Newton and Leibniz in the 17th century and published for the first time in treatises *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus* by Leibniz in 1684 and *Methodus fluxiones* by Newton in a preliminary version in the *Principia* in 1687 [Newton, Principia, p. 252] and in an extended version in 1704 [Newton, Opticks].

The basis of Euler's approach is the concept of function which deliberately makes the demonstrations of the theorems independent of geometrical methods or models.[13] However, before this stage of development, later called arithmetization, of the calculus came out, geometry played the decisive role as a prototype for a mathematical method and, simultaneously, as a prototype for mathematical exactness guaranteeing the reliability (and, in contrast to contingency [Leibniz, Monadology, §§ 30–40], also ensuring the necessity[14]) of the theorems and results [Newton, Quadrature (Harris)], [Leibniz, Specimen]. Moreover, geometry formed the bridge to the *laws of nature* studied by the application of mathematical methods. Newton established a ranking between geometry and arithmetic (algebra) which hampered the self-consistent formulation of the calculus based on arithmetical operations. Newton's approach was not confirmed by the later development by several reasons mainly caused by himself since Newton (i) based the calculus on fluents and fluxions, i.e. on the time variable as a distinguished quantity playing an extraordinary and dominating role and (ii) the new powerful algorithm had been presented in an non-appropriate analytical form caused by the distinguished role of time. Time plays the role of an *independent* variable par excellence, i.e. all other quantities are functions of time, but the time is never a function of any other independent variables. Thus, in conformity with Newton's definition of time as an "absolute mathematical time", the time is also an absolute mechanical quantity. Using the concept of function and the language this concept is represented by Leibniz [Child, p. 26], Johann Bernoulli and developed by Euler [Euler E101] the status of absolute quantities is

[13] The first step out of the realm of geometry was done by the Bernoulli brothers in 1718. "Eine für unser Thema entscheidende Aufgabe JAKOB BERNOULLIS verallgemeinerte das alte Problem der DIDO, und JAKOBS Bruder JOHANN bemerkte, daß dabei die die Bogenlänge betreffenden funktionalen Überlegungen eine nicht mehr konstruktiv erfaßbare, sondern – wie wir heute sagen – transzendente Beziehung unterstellten. Hierauf gründet sich seine 1718 gegebene Definition: Man nennt hier eine Quantität [quantité], die in irgendeiner Weise aus einer variablen Größe [grandeur variable] und aus Konstanten zusammengesetzt ist, Funktion dieser variablen Größe. Diese Definition verleugnet ihre geometrische Herkunft nicht: zum einen erschien in der Analysisfigur noch eine geometrische Einheitsstrecke A, die analytisch sinnlos ist; und zum anderen lag für irgendwelche Funktionen ja kein analytisches Repertoire vor (d.h. keine einschlägige analytische Formeln bergende Schatzkammer), so daß JOHANN BERNOULLI eine beliebig erzeugte Funktion (de quelque manière) nicht anders als durch eine per Hand gezogenen Kurve, also nur geometrisch, darstellen konnte." [Thiele 2007]

[14] Compare Euler on the necessity of the basic relations in mechanics [Euler E015/016, §§ 150–154], the comparison to geometrical truths [Euler E343, Lettre LXXI] and the response of d'Alembert to Euler's derivation of the equation of motion [d'Alembert, Traité].

questioned. However, although Newton implicitly made use of the assumption that *fluents* and *fluxions* $f = f(t)$ are functions of time, he excluded not only the analytical representation of the *inverse* relation $t = t(f)$, but the consideration of this relation in general. The *complete* mathematical and mechanical theory comprising *direct* and *inverse* relations is only formulated for the relation between *fluents* and *fluxions*.[15]

Euler demonstrated that the foundation of the calculus is possible on a basis which is independent of a *geometrical interpretation* of the results. By this procedure, Euler recovered that part of the Leibnizian foundation which had been called the *calculus of differences and sums* by Leibniz and which was commonly represented by the notions (and names) of *differentials* and *differentio-differentials* or differences of first and second order without any limitation in the order [l'Hospital],[16] [Wolff, Math Lexicon], [Châtelet, Institutions]. In the Preface of *Institutiones*, Euler emphasized that the treatment is completely analytically and no figures are needed for the derivation or interpretation.[17]

From Euler's introductory remarks it follows that there was still a need of the foundation of the calculus in 1755. Euler intended to "derive the whole differential calculus from true principles" [Euler E212, Preface].[18]

The basic notions for the interpretation of the calculus are related to the distinction between *assignable* and *unassignable* quantities which had been invented by Leibniz simultaneously for the interpretation of geometrical and arithmetical models

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[15] "Calculate the fluxions if the fluents are given and, calculate the fluents if the fluxions are given" [Newton, Method of Fluxions] (compare Chaps. 2 and 3). The representation of the path as a function of time, $x = x(t)$, and, vice versa, the inverse relation, the time as function of path, $t = t(x)$, had been only invented and analytically formulated by Euler [Euler E015/016, §§ 1–152]. Newton did not consider time as function of another independent variable. Euler based the representation on the formulas expressed in terms of infinitesimal and finite quantities, i.e. the path element dx, the time element dt and the velocity v, respectively, (i) $\mathrm{d}x = v/\mathrm{d}t,\ x = x(t)$ and (ii) $\mathrm{d}t = \mathrm{d}x/v,\ t = t(x)$ [Euler E015/016, §§ 37–55]. As a consequence, the change of velocity is either a function of time, $dv \sim dt$, or a function of translation, $dv \sim ds$ [Euler E015/016, §§ 150–154].

[16] The concept of function is missing in l'Hospital [Thiele 2007].

[17] Lagrange continued and developed Euler's approach in mechanics and mathematics in goal and spirit as far as the analytical or geometry independent demonstrations are concerned [Lagrange, Mécanique].

[18] "Ich habe mir daher vorgenommen, im gegenwärtigen Werke die ganze Differentialrechnung aus wahren Prinzipien abzuleiten und so ausführlich abzuhandeln, daß ich von dem bis jetzt darin erfundenen nichts überginge. Ich habe dasselbe in zwei Teile geteilt. In dem ersten beweise ich zuvörderst die Hauptregel der Differentialrechnung und zeige dann, wie man von allen Arten der Funktionen sowohl einer als auch mehrerer veränderlicher Größen nicht nur die ersten, sondern auch die Differentialien aller übrigen Ordnungen findet. In dem anderen beschäftige ich mich mit den Anwendungen dieses Kalküls, teils in der Analysis des Endlichen, teils in der Lehre von den Reihen und setze dabei vorzüglich die Lehre vom Größten und Kleinsten (de Maximis et de Minimis) auseinander. Von seinem Nutzen in der Geometrie aber rede ich nicht, weil man darüber Werke genug hat, da sogar die ersten Prinzipien der Differentialrechnung aus der Geometrie hergenommen, und dieselbe auf diese Wissenschaft gleich nach ihrer ersten Entwicklung mit der größten Sorgfalt angewandt worden sind. Das gegenwärtige Werk hält sich durchaus innerhalb der Grenzen der reinen Analyse, so daß ich auch nicht einmal eine einzige Figur zur Erläuterung nötig gehabt habe." [Euler E212, Preface, (Michelsen)]

for the relation between finite lengths, areas and volumes and infinitesimal lengths, areas and volumes as well as finite real numbers and infinitesimal numbers (compare Arthur [Arthur, Syncategoremtic][19]). The distinction between geometrical and arithmetical representation had not been applied to determine the difference between the notions of "unassignable geometrical quantities" and "unassignable numerical quantities". Geometrically, a point is at least an unassignable because unextended object with respect to extended geometrical objects like lines, areas and solids.

Euler based mechanics on analytical method and the corresponding definition of mechanical quantities on arithmetical relations. This approach necessarily results in a distinction between the two kinds of "unassignable quantities" which had to be related to two corresponding kinds of "assignable quantities". The difference between "assignable" and "unassignable" has to be solely defined using arithmetical operations addition, subtraction, multiplication and division. Hence, Euler invented a new approach in the foundation of mathematics which had been later summarized as the "arithmetization of mathematics" by Felix Klein [Klein, Arithematization]. The unassignable quantity is determinate with respect to a "chosen assignable quantity" ("vorgegebene Größe")[20] [Klein, Elementarmathematik]. However, the implicit assumption made by Leibniz and his followers is that the chosen quantity as small it may ever be is assumed to be different from zero.[21] In contrast to most of his pre-

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[19] Following Arthur, Leibniz developed the interpretation of the differentials as fiction not only in his answer to Nieuwentijt in the 1690s, but much earlier in 1676 [Arthur, Fictions]. Hence, Newton, Leibniz and Euler invented the foundations and the interpretations of the calculus in analyzing the general relation between the *continuum* and *discrete* quantities. Then, the *mechanical* interpretation of the *continuity of motion* is correlated with the arithmetical representation of motion by *discrete temporal* and *spatial intervals* of different magnitude (compare Chaps. 1, 2, 4 and 6).

[20] "In this paper I attempt to throw light on these issues by exploring the evolution of Leibniz's early thought on the status of the infinitely small in relation to the continuum. The picture that emerges differs in one way or another from all those detailed in the previous paragraph. For one can distinguish among Leibniz's early attempts on the continuum problem three different theories involving infinitesimals interpreted as non-Archimedean magnitudes. (i) the continuum consists of assignable points separated by unassignable gaps (1669); (ii) the continuum is composed of an infinity of indivisible points, or parts smaller than any assignable, with no gaps between them (1670–71); (iii) a continuous line is composed of infinitely many infinitesimal lines, each of which is divisible and proportional to an element (conatus) of a generating motion at an instant (1672–75). By early 1676, however, he has already reached the conclusion that (iv) infinitesimals are fictitious entities, which may be used as *compendia loquendi* to abbreviate mathematical reasonings; they serve as a shorthand for the fact that finite variable quantities may be taken as small as desired, and so small that the resulting error falls within any preset margin of error. Thus on the reading I propose here, Leibniz arrived at his interpretation of infinitesimals as fictions already in 1676, and not in the 1690's in response to Nieuwentijt's and Rolle's criticisms, whatever may have been his later hesitations." [Arthur, Fictions]

[21] Arthur summarized Leibniz's approach in terms of an unlimited approximation. "They serve as a shorthand for the fact that finite variable quantities may be taken as small as desired, and so small that the resulting error falls within any preset margin of error." [Arthur, Fictions] Arthur claimed that Leibniz introduced this interpretation much earlier than it is commonly thought. "For a good Statement of this position, see in particular D. M. Jesseph, 'Leibniz on the Foundations of the Calculus: the Question of the Reality of Infinitesimal Magnitudes', Perspectives on Science 6: 1 & 2 (1998), pp. 6–40: 'the fictional treatment of infinitesimals clearly appears designed in

decessors[22] and most of his contemporaries,[23] Euler included at first (i) the number zero into the consideration of the unassignable quantities and, at second (ii) the infinite as a number which can be increased as any other number [Euler E387/388]. Hence, Euler went beyond the frame which had been established by geometrical models (like horn angles, characteristic triangle, the generation of lines by the motion of a point [Newton, Quadrature (Harris)][24]). Newton twofold excluded Cavalieri's model of indivisibles, at first, he denied that the mathematical quantities can be considered as "composed of parts extremely small" and, at second, he denied also the validity of the inverse operation that the mathematical quantities can be "decomposed into parts extremely small". Both operations are rooted in arithmetics. Hence,

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response to them [Wallis and Bernoulli] and to the critics of the calculus [Nieuwentijt and Rolle]. If I am right we can see this doctrine take shape through the 1690s as Leibniz tries to settle on an interpretation of the calculus that can preserve the power of the new method while placing it on a satisfactory foundation.' Cf. also Detlef Laugwitz: 'It was not before 1701 that Leibniz was forced to clarify his opinions, both mathematically and philosophically, on the use and nature of infinitesimals': 'Leibniz' Principle and Omega Calculus', pp. 144–154 in *Le Labyrinth du Continu*, ed J-M. Salanskis and H. Sinaceur, Paris, 1992, p. 145." [Arthur, Fictions]

[22] Galileo stated that the set of the squares of integers is made up by the same number of elements as the set of integers [Galileo, Discorsi]. Hence, the infinite can be defined by different kinds of numbers. Newton discussed infinite numbers of different magnitude [Newton, Notebook] (compare Chap. 3).

[23] Nieuwentijt confined possible orders defined by differentials to the ordering determined by the differential and the differentio-differentials [Nieuwentijt, Analysis]. For a contemporary discussion of Nieuwentijt's approach compare [Arthur, Syncategorematic].

[24] In the original version of the model it is accentuated that, although the line is generated by the motion of a point, the "extension" is only determined, i.e. assignable, in the direction of motion, but remains to be *unassignable* perpendicular to that direction. The subsequent motions of the line generating an area and the motion of an area is generating a solid (plenum) removing subsequently the unassignable and unassigned parts of the objects and substitute these parts by assignable objects. This process of subsequent replacements had been discussed by Newton [Newton, Quadrature]. Hence, Newton is aware of the problem to transform an unassignable object, a point, into an assignable object and correlated, on contrast to Cavalieri who assumed indivisibles the line is made up, the *replacement* of point by line, lines by areas and areas by solids, which is described in terms of time related generation being different from an instantaneous creation or annihilation of objects (in case of transcreation [Leibniz, Hypothesis], [Leibniz, Pacidius]). The original version of both types of processes is the ancient prototype due to Heron. "Punkt ist, was keinen Teil hat oder eine dimensionslose Grenze oder eine Grenze einer Linie, und sein Wesen ist es, nur dem Gedanken faßbar zu sein, weil er sowohl ohne Teile als auch ohne Größe ist. Man sagt daher, daß er von derselben Beschaffenheit ist, als das Jetzt in der Zeit und die im Raume als Stelle festgelegte Einheit. (...) denn aus der Bewegung des Punktes oder richtiger aus der Vorstellung eines im Fluß befindlichen Punktes entsteht die Vorstellung einer Linie, und in diesem Sinne ist der Punkt Anfang der Linie wie die Fläche der des soliden Körper." Tropfke referred to Heron: "Eine Linie ist eine Länge ohne Breite und Tiefe oder das, was innerhalb der Größe zuerst Existenz annimmt, oder, was nach einer Dimension Ausdehnung hat und teilbar ist; sie entsteht, indem ein Punkt von oben nach unten gleitet gemäß der Stetigkeit und ist eingeschlossen und begrenzt durch Punkte, während sie selbst Grenze einer Fläche ist." [Tropfke, pp. 29–31] Tropfke continued with a reference to Euclid: "Eine Linie ist eine Länge ohne Breite." [Euclid, Elements, I, Def. 2] The infinitesimal translation can only be assigned to a line as an infinitesimal translation in the direction of the line labelled by dx being different from zero. Any translation perpendicular to the line is impossible, i.e. $dy = 0,\ dz = 0$.

the arithmetical procedures are completely excluded or, in case of any rudimental relicts, are replaced with geometrical objects and methods.[25] Newton also paved the way for an arithmetization by the invention of fluents, fluxions and moments, but, having performed the calculations he reinvented geometry to confirm the reliability of the demonstration and to present the results.

5.2 Euler's Foundation of the Calculus

Leibniz based the calculus on geometry and arithmetics. The geometry related model of a curved line in a plane is also transformed into an arithmetical version using the notion of function [Bos]. However, Leibniz did not make use of the idea of a function from the very beginning, but only later in 1694 [Child, p. 26]. Euler, on the contrary, becomes familiar with the notion of function in the very beginning of his scientific career due to the education by his teacher Johann Bernoulli. Not surprisingly, Euler did not only contribute in numerous papers to the development of the concept of function, but invented also the currently used notation of functions by the symbol $f(x)$ in 1734 [Euler E044][26] distinguishing between variables, constants, independent and dependent variables [Euler E101]. In 1727 Euler decided to give an arithmetical foundation of the calculus. Hence, Euler was confronted with the problem to invent a full (unrestricted) representation of all arithmetical operations for infinitesimal quantities. The notion of "infinitesimal quantity" had been introduced in a non-geometrical frame being related to the numerical difference between infinitesimal and finite. The commonly accepted version of the relation between infinitesimal and finite quantities was based on the idea that a finite quantity can be represented by the "product" of an infinitesimal and infinite quantity introduced by Galileo (compare the comment of Leibniz [Leibniz, Specimen, I (6)]). The living force results from an infinite number of non-interrupted succession of impressions of dead force.

> (...) vis est viva, ex infinitis vis mortuae impressionibus continuatis nata. Et hoc est quod Galileus voluit, cum aenigmatica loquendi ratione percussionis vim infinitam dixit, scilicet, si cum simplice gravitates nisu comparetur. [Leibniz, Specimen, I (6)]

Numerically, this relation discussed by Galileo and Leibniz is represented either (i) by the infinite sum of the "impressions of the dead force" or (ii) by product of an "in-

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[25] "I don't here consider Mathematical Quantities as composed of parts extremely small, but as generated by a continual motion (...). And after this manner the Ancients by carrying moveable right Lines along immoveable ones in a Normal position or Situation, have taught us the Geneses of Rectangles." [Newton, Quadrature, (Harris)] The representation of the geometrical objects is incomplete or only implicitly given since in contrast to the ancient prototype [Tropfke], the "extension" of the line perpendicular to its direction is not explicitly analyzed.

[26] Beside the symbol for the function, Euler invented the symbols for e and pi [Euler E015/016] in 1736, $\sin x$ and $\cos x$ [Euler E101/102] in 1748, Σ, Δ, Δ^2 [Euler E212] in 1755 and $i = \sqrt{-1}$ [Euler E671] in 1777 (published in 1794). (compare Thiele, *Leonhard Euler, the Decade 1750–1760* [Bradley, D'Antonio, Sandifer])

finitesimal impression" and an "infinite number" if the infinitesimal impressions are of *equal* magnitude. The equality of two of infinitesimal impressions is undoubtly a numerival relation between infinitesimal quantities. Following Euler, this model for the relation between finite, infinitesimal and infinite "objects" ("eine unendlich grosse Menge unendlich kleiner Dinge, eine endliche Größe darstellen könnte") was accepted and well-known in higher mathematics where the "infinite large and small is commonly considered in such way".

> Der Herr von *Leibniz* behauptet, dass die Anzahl der einfachen Dinge, welche einen Körper ausmachen, unendlich gross sei, und da liesse sich noch einigermassen begreifen, wie ein unendlich grosse Menge unendlich kleiner Dinge, eine endliche Grösse darstellen könnte: da in der höheren Mathematik das unendlich grosse und das unendlich kleine auf diese Weise betrachtet zu werden pflegt, welche auch vielleicht dem Herrn von *Leibniz* zu diesen Gedanken mag Anlass gegeben haben. [Euler E081, § 60]

All expressions are either of finite or of infinitesimal magnitude where the latter are distinguished in magnitude according to the order of smallness they belong to, i.e. the 1st, 2nd, 3rd &c. order. The quantities belonging to consecutive orders are simultaneously either "incomparable small" ("incomparabiliter minus") or "incomparable large" ("incomparabiliter maius") if being compared to a quantity of higher order or lower order, respectively.[27] The rejection of terms being "incomparable small" in analytic expressions is introduced as a distinct numerical procedure where the terms depending on differentials of higher degree are taken to be zero or, infinitesimal with respect to all other homogeneous terms of the same magnitude. Obviously, the *inverse* relations are also valid since a term *greater* than the infinitesimal term is necessarily either finite or infinite. Hence, Euler said that the "infinitesimal quantities are really zero with respect to any finite quantity" [Euler E212, Preface], but accentuated the need for being aware that this theorem is only valid for the "arithmetical", but not also for the "geometrical" ratio of infinitesimal and finite quantities [Euler E212, § 85].

Newton invented the same procedure which had been later criticized by Berkeley [Berkeley, Analyst]. The main objection was that the differentials are "sometimes" different from zero and "sometimes" equal to zero.[28] The same problem appeared in the Leibnizian version of the calculus. In 1691/92, Johann Bernoulli formulated an arithmetical rule to remove this ambiguity which provides an algorithm to handle Leibniz's "evanescent quantities", i.e. $a \pm o = a$ [Johann Bernoulli, 1691/92 (Juschkevich)]. This idea is closely related to the interpretation

[27] Usually, only the relation of infinitesimal to finite quantities represented by dx and x is discussed where "dx is *incomparably small* in relation to x", but not also the inverse relation "x is *incomparably large* in relation to dx".

[28] "Let now the Increments vanish, and their last Proportion will be 1 to nx^{n-1}. But it should seem that this reasoning is not fair or conclusive. For when it is said, let the Increments vanish, i.e. let the Increments be nothing, or let there be no Increments, the former Supposition that the Increments were something, or that there were Increments, is destroyed, and yet a Consequence of that Supposition, i.e. an Expression got by virtue thereof, is retained. Which, by the foregoing Lemma, is a false way of reasoning. Certainly when we suppose the Increments to vanish, we must suppose their Proportions, their Expressions, and every thing else derived from the Supposition of their Existence to vanish with them." [Berkeley, Analyst, §§ XII and XIII]

of infinitesimal quantities as “fictions” [Arthur, Fictions]. The source of the controversies is mainly the ambiguity in the definition of “infinitesimal quantities”. Newton invented a “motion related” model where the quantities are generated by a process. Hence, the idea of increments or decrements represented by indivisibles [Cavalieri] is formally rejected, but basically preserved in the reckoning procedure, however, only in rudimentary form. Leibniz made use of the scheme of differences and also excluded the interpretation in terms of increments since an “evanescent” or “fictitious” quantity cannot be able to cause an augmentation or a diminution of any finite quantity. Bernoulli’s rule established in 1691 perfectly fits for this arithmetical operation, i.e. using the former terminology, for the “arithmetical ratio” [Euler E212]. In 1694, Nieuwentijt completed the algorithms for the case of the “geometrical ratio”, i.e. the division of a finite number by an infinitesimal number and a finite number by an infinite number [Nieuwentijt, Analysis].[29] The crucial point is the interpretation of the status of “infinite numbers” and of infinite numbers of different magnitude corresponding to infinitesimals of different magnitude [Nieuwentijt, Analysis]. The problem to clarify the status of the infinite in order to shed a light on the status of infinitesimals is behind all attempts of the invention and

[29] “In the *Considerationes* of 1694 Nieuwentijt criticized the first lemmas in Newton’s *Principia* in which it is supposed that infinitesimals may be disregarded, because this supposition leads to absurdities (pp. 9–15). Moreover he questioned the validity of the procedure Newton follows to obtain the basic rules of differentiation (pp. 24–27). However, the bulk of Nieuwentijt’s mathematical work is devoted to proving that the use of higher-order infinitesimals (such as $(\mathrm{d}x)^2$, $\mathrm{d}x\mathrm{d}y$), and particularly the way in which Leibniz employs them, leads to contradictions and is dangerous from a religious point of view. Adoption of higher-order infinitesimals Nieuwentijt regarded as dangerous, in that it might lead to the assumption that man can grasp something of the infinite. This would obscure the fact that while it was our Creator’s will that we are created in such a way that, although our comprehension can show us a quantity which is greater or smaller than whatever perceived quantity, we are still only able to perceive finite and determined objects; the human intellect is not capable of rising to a true and adequate understanding of the infinite itself (*Analysis infinitorum* (1695), praefatio p. 4). Leibniz is criticized for not taking into account the unbridgeable gulf between the finite and the infinite, in that he ascribes finite qualities to higher-order infinitesimals: many contradictions ensue from this presupposition.

Another reason for Nieuwentijt’s rejection of higher-order infinitesimals is that employing them presupposes that matter (finite extension) is divisible into infinitely small parts, which can be subdivided again etc. without ever reaching the point of being annihilated, suggesting that matter is indestructible, and therefore eternal (which for Nieuwentijt is the same as atheism). He believes that his own axiom, according to which anything that is multiplied by an infinite quantity and does not become a magnitude is a mere nothing (thus excluding the use of higher-order infinitesimals), avoids this danger while it expresses the fundamental truth that any quantity can be reduced to nothing by an infinite power, and conversely, that any quantity can be reduced to nothing, which points to the infinite power of God and the transitoriness of all matter. In the *Considerationes* Nieuwentijt even expressed the hope that this axiom ‘would provide us with an invincible argument (...) against the eternity of the world and other despicable dogmas of wretched atheism, and would defend the (...) power of the Creator of such great things against the blasphemous rages of the (...) philosophers, through an argument developed from creation and in conformity with revelation’ (*Considerationes*, pp. 38–39).” [Vermeulen] foundation of the calculus by Newton and Leibniz and, later, also in the criticism of Nieuwentijt, Cantor [Cantor][30] and Weierstraß.[31]

Euler contributed to the formulation and the solution of the problem in books and papers written between 1727 and 1770, (i) *Calculus differentialis* 1727 (published only in 1983), (ii) *Institutiones calculi differentialis* [Euler E212] 1755 and (iii) *Vollständige Anleitung zur Algebra* [Euler E387] 1770. However, the complete version of the early manuscript on *Calculus differentialis* had not been published until now [Euler 1727].[32] The foundation of the calculus is formulated as an analytical problem to determine the relations between *finite* and *infinitesimal increments* of finite variables and *functions* of finite variables. As the variables, the increments are quantities of their own origin and rules, i.e. the magnitudes of the increments are *independent* of the magnitudes of the variables.

From the very beginning, Euler explained the notion of differential making use on the notion of function.

> The *differential calculus* teaches us to determine the increment of any function given from the infinitely small increments of the quantities composing it. These infinitely small increments are called *differential*. To *differentiate* a quantity means to determine its differential. [Euler 1727][33]

Euler continued to explain the arithmetically determined distinction between the *calculus of differences* and the *calculus of differentials* as being caused by and arising from the difference between *indeterminate* and *determinate* quantities, respectively. A finite increment can be taken to be of *arbitrary* magnitude whereas the infinitesimal increment is a determined quantity since it is "taken as infinitely small", i.e. the Bernoulli criterion is fulfilled. An infinitesimal quantity can neither increase nor decrease any of the finite quantities.

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[30] Although Cantor invented the analysis of transfinite number, he was eager to banish infinitesimals and referred to Leibniz who claimed that the differentials are fictitious quantities. "(...) und darum schon von *Leibniz* als bloße *Fiktionen* charakterisiert werden, z. B. In der Erdmannschen Ausgabe, S. 436), (...)." [Cantor]

[31] "It is perhaps only fair to point out that some of Euler's works represent outstanding examples of eighteenth-century formalism, or the manipulation, without proper attention to matters of convergence and mathematical existence, of formulas involving infinite processes. He was incautious in his use of infinite series, often applying to them laws valid only for finite sums. Regarding power series as polynomials of infinite degree, he heedlessly extended to them wellknown properties of finite polynomials. Frequently, by such careless approaches, he luckily obtained truly profound results (...)." [Eves, p. 435]

[32] Leonhard Euler, Calculus differentialis. Manuskript, 30 Seiten. Archiv der Petersburger Akademie, f. 136, op. 1, Nr. 183. [Beschrieben bei A. Juschkevic, Euler's unpublished manuscript Calculus Differentialis. In: Euler-Gedenkband des Kantons Basel. Basel: Birkhäuser 1983, S. 161–170]

[33] "Docet igitur Calculus differentialis functionis cujuscunque incrementum invenire ex datis quantitatum eam ingredientium incrementis infinite parvis. Haec incrementa infinite parva vocantur differentialia. Et quantitatem differentiare significat ejus differentiale invenire." [Euler 1727]

> It is evident that the differential calculus is a special case of the calculus which I have enunciated above, since what was before assumed arbitrary is now taken to be infinitely small. [Euler 1727][34]

In contrast to Leibniz who preferentially considered the differentials as differences (compare Chap. 3), Euler stressed that the differential calculus is a calculus of increments[35] (compare the review by Lacroix [Lacroix]). Nevertheless, also Leibniz emphasized the relation between the differential calculus and the calculus of differences and sums.[36]

> 1. Ex iis qua de calculo differentiarum finitarum prolata fieret, facile erit intelligere, quid fit calculus differentialis seu differentiarum infinite parvarum. [Euler 1727, Caput 2, § 1]

The foundation of the calculus is the reliable basis for the development of mechanics or the science of motion analytically demonstrated, i.e. the famous *Mechanica sive scientia motus analytice exposita* written between 1734 and 1736 [Euler E015/016, Mechanica]. Obviously, Euler mirrored Newton's development between 1665 and 1687 who at first invented the *Method of Fluxions* which had been subsequently applied to mechanics.

However, there is an essential difference between Newton's and Euler's foundation as far as the *generation* and *formation* of the infinitesimal quantities is

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[34] "Perspicuum est Calculum differentialem, ejus, quem ante exposui, calculi esse casum specialem: nam, quod ibi quantumvis erat assumtum, hic ponetur infinite parvum." [Euler 1727]

[35] "If radical quantities increase by finite quantities, the calculus might be called the *calculus of finite differences* or *finite increments*. It should not be confounded with the calculus in which radical quantities can assume infinitely small increments. This case will be discussed later on, after the exposition of the calculus of finite differences. To the best of my knowledge nobody yet has developed the latter, so that it might seem unnecessary to use it for the study of the rules of differential calculus. However, when I saw that quite a lot scholars including those not unexperienced in this higher analysis, do not have any correct, or even frequently possess a false, idea about the calculus of differences I thought it inappropriate to omit this calculus and decided to explicate it before going over to the differential calculus." [Euler 1727] Juschkevich commented: "Euler's remark to the effect that 'nobody yet has developed calculus of finite differences' should not be explained away by his fail known his predecessors I. Newton (1711), B. Taylor (1715), F. Nicole (1720), P. de Montmort (1720) and others. Even during his years of study Johann I Bernoulli in Basle, i.e. before his departure for Petersburg in 1727, Euler got used to keep a close watch on all the available mathematical literature, and the works of the above-mentioned savants should have known to him. However, these savants applied the method of finite differences to the theory of interpolation, to the calculation of differences and sums of generalized power (...) and its reciprocal quantity, and, later on, to recurrent series introduced at the time by A. de Moivre etc., whereas Euler, in his manuscript, wanted to describe the algorithm calculus of finite differences so as to use it for the most simple class functions which he enumerated in the beginning of this work. Exactly that of such a System of rules in the available literature is what Euler is spook about in the passage just quoted." [Juschkevich, Euler] For Euler's approach to mathematics compare also Knobloch [Knobloch, Notizbücher].

[36] "When my infinitesimal calculus, which includes the calculus of differences and sums, had appeared and spread, certain over-precise veterans began to make trouble; (...). For I have, beside the mathematical infinitesimal calculus, a method also for use in Physics, (...) and both of these I include under the Law on Continuity (...). I take for granted the following postulate: *In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may be also included*." [Leibniz, (Child), p. 145]

concerned. Newton assumed the generation of mechanical quantities by a non-arithmetical procedure, i.e. by a *continuous* flux modelled by the "flow of time". Hence, the differentials are bound to their origin as genuinely continuously processing while generating the continuously growing increments. Euler, on the contrary, considered increments of invariant "size" and magnitude independently of their absolute magnitude. In case of finite size increments, the total increment is not represented by a *continuous growth*, but by a *discontinuous* augmentation or diminution being analytically asserted by the sum or a multiple of discrete elements. Although for the *differential* calculus the increments are of *infinitesimal* magnitude, their discrete nature is preserved so that any total increment is composed of a multiple of discrete elements and also asserted by a sum. Hence, the only arithmetical operations to introduce the representation of independent variables being composed of variables and increments are addition and subtraction. Hence, Euler did not only pave the way for an *arithmetization* of mathematics and mechanics, but simultaneously for a *discretization* of *some* of the *mechanical* quantities like time, space and velocity. In view of 20th century physics, there is an essential distinction between discretization and quantization. Quantization concerns only those mechanical quantities which are excluded from the discretization according to Euler's procedure like energy, mass, charges, forces whereas time and space are discretisized, but not quantized.

5.2.1 Calculus Differentialis: Finite and Infinitesimal Increments

In 1727, Euler developed the almost complete theoretical background being available in that time to formulate all later versions of the foundation of the calculus which had been implicitly published in the *Mechanica* in 1736 and other papers on mechanical problems and, explicitly published in *Institutiones calculus differentialis* [Euler E212] and *Algebra* [Euler E387] published in 1750 and 1770. The crucial point is to define precisely the relations between quantities of infinitesimal, finite and infinite magnitude.

> At the same time Euler (...) warned against errors which may happen when the differentials or infinitesimals are omitted before the given expression 'is completely prepared' *('quam expressio penitus est adornata')*. He also explained the rules for the omission of infinitely small of all higher orders from expressions containing infinitely small of various orders, and introduced infinitely large quantities of various orders (reciprocals of corresponding infinitely small). Euler accompanies all this by numerous examples. [Euler 1727 (Juschkevich)]

Only almost thirty years later, the complete set of rules to handle correctly the "completely prepared" expressions is presented in the *Institutiones* [Euler E212] in 1755. However, reading Euler's comments it becomes clear that the dispute on the foundation of the calculus was not terminated, but still progressing. Meanwhile, the idea of limits had been invented by d'Alembert and Lagrange.

In the treatise *Institutiones calculi differentials* [Euler E212] and its earlier version *Calculus differentialis* [Euler 1727], Euler assumed that the increments ω of an independent variable x are given by an arithmetical progression, i.e. the increase of increments is not limited by any additional condition for truncation.

> 1. Ex iis, quae in Libro superior de quantitatibus variabilibus atque functionibus sunt exposita, perspicium est, prout quantitas variabilis actu variatur, ita omnes eius functiones variationem pati. Sic, si quantitas variabilis x capiat incrementum ω, ita ut pro x scribatur $x+\omega$, omnes functiones ipsius x, cuiusmodi sunt xx; x^3; $\frac{a+x}{xx+aa}$, alios induent valores: scilicet abibit in $xx+2x\omega+\omega\omega(\ldots)$ transmutabitur.
>
> 2. Quae cum sint satis exposita, propius accedamus ad eas functionum affectiones, quibus universa analysis infinitorum innititur. Sit igitur y functio quaecunque quantitatis variabilis x: pro qua successive valores in arithmetica progressione procedentes substituantur, scilicet: x; $x+\omega$; $x+2\omega$; $x+3\omega$; $x+4\omega$; & c. ac denotes y^I valorem quem functio y induit, si in ea loco x substituantur $x+\omega$; $(\ldots)$. [Euler E212, §§ 1 and 2]

Hence, the mathematical foundation may be considered as reliable as any other mathematical discipline which is rooted in the *ordered* set[37] of *integers* 1, 2, 3,... including the *arithmetical* operations defined for this type of numbers.[38]

$$x, x+\omega, x+2\cdot\omega, x+3\cdot\omega, \ldots \tag{5.1}$$

Euler made use of the following self-explaining labelling of variables and functions where all ordering is related to the ordered set[39] of integers [Euler E212, § 2] and the *series of terms* generated by the increments are correlated to a *series* of terms made up by the values of the *function* belonging to different arguments.

$$x; \quad x+\omega; \quad x+2\omega; \quad x+3\omega; \quad x+4\omega; \tag{5.2}$$

$$y; \quad y^I; \quad y^{II}; \quad y^{III}; \quad y^{IV}; \quad y^V; \tag{5.3}$$

Since the arithmetical series can be continued to infinity, the series of the values of the function can be also proceeded to infinity.

[37] Obviously, Euler did not made use of the notion of a "set". However, the *ordering procedure* based on integers is an essential part of the mathematical representation of mechanical relations, e.g. the equality of space and time intervals in case of *uniform motion* (compare Euler's comments on space and time [Euler E149]).

[38] Euler's choice of the name for the increment may provide us with a most elucidating insight in the subtle and supreme procedure being full of irony in handling the language and insight into the problem Euler referred to predecessors and contemporaries. Euler simply replaced the small *Latin* letter "*o*" chosen by Newton for labelling the "*infinitely* small quantity" with the small *Greek* letter "ω" for labelling the "*finite* increment" of the variable x. Then, the "letter *o*" becomes a part of the name "omega" anticipating in the denotation the procedure established later where the "differential calculus" becomes a special case of the "calculus of finite differences".

[39] Thereby, the Leibnizian idea of space and orders as different orders of things [Leibniz, Initia] are not only *qualitatively*, but also *quantitatively* represented which may be considered as a merging of Leibniz's and Newton's concepts of space and time (compare Chap. 2).

> 3. Quemadmodum series arithmetica x; $x+\omega$; $x+2\omega$; &c. in infinitum continuari potest, ita series ex functione y orta y; y^I; y^{II}; &c. quoque in infinitum progredietur, eiusque natura pendebit ab indole functionis y. [Euler E212, § 3][40]

The investigation of the magnitude is performed using the "general term" [Euler E212, § 3] which is currently called the function, e.g. $y=\dfrac{a}{bx+c}$. The increment Δy of the function is called "difference" [Euler E212, § 4],[41] the first, second and all differences of higher degree are calculated by the application of the same algorithm [Euler E212, § 5]

$$y^I - y = \Delta y, \quad \Delta y = y^I - y, \quad \Delta y^I = y^{II} - y^I \tag{5.4}$$

There is a different number of intervals (or measurements in case of a body travelling non-uniformly) needed for the calculation of the differences of different order [Euler E212, §§ 10ff.].

One interval

$$\Delta y = y^I - y \tag{5.5}$$

Two intervals

$$\Delta\Delta y = y^{II} - 2y^I + y \tag{5.6}$$

Three intervals

$$\Delta^3 y = y^{III} - 3y^{II} + 3y^I - y \tag{5.7}$$

with

$$\Delta\Delta y = \Delta y^I - \Delta y, \qquad \Delta\Delta y^I = \Delta y^{II} - \Delta y^I \tag{5.8}$$

The general scheme of differences [Euler E212, § 7] for a function $y=f(x)$ is represented as follows

$$\begin{array}{cccccc}
x & x+\omega & x+2\omega & x+3\omega & x+4\omega\ldots & \&\text{c.} \\
f(x) & f(x+\omega) & f(x+2\omega) & f(x+3\omega) & f(x+3\omega),\ldots & \&\text{c.} \\
f & f^I & f^{II} & f^{III} & f^{IV},\ldots & \&\text{c.} \\
 & \Delta f & \Delta f^I & \Delta f^{II} & \Delta f^{III},\ldots & \&\text{c.} \\
 & & \Delta\Delta f & \Delta\Delta f^I & \Delta\Delta f^{II} & \Delta\Delta f^{III},\ldots\&\text{c.} \\
 & & \Delta^{(3)} & \text{f } \Delta^{(3)} f^I & \Delta^{(3)} f^I I,\ldots\ \&\text{c.} &
\end{array} \tag{5.9}$$

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[40] Euler accentuated the "innate character or inborn quality of the function y", i.e. the inherent properties of the function being independent of the magnitudes of the argument and the increment [Euler E212, § 4].

[41] Euler mentioned that his approach is free of the limitations known from the investigation of series since an unlimited increase and decrease of the function is possible. "In doctrina quidam serium sumi solet $\omega=1$; verum hic ad nostrum institutum expedit, valore generali uti, qui pro arbitrio augeri diminuive queat." [Euler E212, § 4]

Euler demonstrated the application of the algorithm for simple examples [Euler E212, § 13].

$$y = x^2,\ y = (x+\omega)^2,\ \Delta y = 2\omega x + \omega\omega,\ \Delta\Delta y = 2\omega\omega,\ \Delta^3 y = 0,\ \Delta^4 y = 0; \&c. \tag{5.10}$$

Here, the advantage of the calculus of differences becomes apparent and can be easily grasped. The relations between differences of different degrees are *analytically completely* defined by their dependence on the variable x and the increment ω. Moreover, the terms stemming from different positions in the general scheme are *uniquely* distinguished by their *analytical* representation.[42] As a consequence of the introduced rules, there are no terms in the representation of the 1st difference Δy depending only on the variable x and being independent of the increment. In the simplest case it follows $\Delta y = x + \omega - x = \omega$. This result is independent of the magnitude of the variable as well as of the magnitude of the increment. Hence in the general case, the increment Δy of the function y depends on (i) the magnitude of the variable x and (ii) on the magnitude of the increment ω, or, the increment $\Delta y = F(x, \omega)$ is a function of two variables since the magnitude of the increment can be also arbitrarily modified.[43]

Accentuating the common features of the algorithms,[44] Euler made use of the notation Leibniz had introduced for the differentiation (e.g. $d\overline{xx} = 2xdx$ [Leibniz, Nova methodus], compare Chap. 3) also for the calculus of finite differences to demonstrate the effect of an operator "Δ." being not applied to a "function" $y = y(x)$, but directly to an algebraic "expression" the function is made up.

$$\Delta y = 2\omega x + \omega\omega \text{ (a)} \quad \Delta.xx = 2\omega x + \omega\omega \text{ (b)}. \tag{5.11}$$

The increment is generated by the direct application of the operation Δ to the quantity x [Euler E212, § 13], i.e. the increment is not arbitrarily chosen, but results from a special operation which is defined within the frame of the calculus of *finite* differences.

$$\Delta.\ x = \omega \qquad \Delta\Delta.\ x = \Delta\omega = 0 \tag{5.12}$$

There is no increment of the increment since the quantity ω is assumed to be constant.

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[42] This will be of great importance for the *mechanical interpretation* of different terms (compare Chaps. 4, 6, 7 and 8). The various mechanical notions are analytically quite differently represented. Therefore, a mixing up of their origin and their role with those of other terms is excluded.

[43] The magnitude of the increment ω is indeterminate, i.e. the ratio of the variable x and the increment ω is also *indeterminate*. Later, Euler claimed that, on the contrary, the relation between a finite value of the variable x and an infinitesimal increment is always *determinate* independently of the magnitude of x since it holds $x + n\ dx = x$.

[44] "If his rival had known of these things, he would not have used dots to denote the degree of the differences, which are useless for expressing the general degree of differences, but would have used the symbol d given by our friend or something similar, for then d^e can express the degree of the difference in general. Besides everything which was once referred to figures, can now be expressed by the calculus." [Leibniz, (Child) p. 54]

$$\Delta.\ xx = 2\omega x + \omega\omega \tag{5.13}$$

The application to mechanics is straightforwardly. The operation $\Delta.$ does not modify the mechanical meaning of the variable. Hence, the increment of a length is also a length, the increment of a distance is also a distance &c. Following Newton and expressing the distance l^I instead of $l^I = l + \lambda$ by fluents and fluxions, i.e. by increments of time and extra quantity is needed $l^I = l + \dot{l} \cdot o$. Then, it is implicitly assumed that time and the increments of time is represented by $t^I = t + \dot{t} \cdot o$ instead of $t^I = t + \tau$ and, by the same reason, an extra quantity $\dot{t}$ is required. As such an additional condition, Newton assumed $\dot{t} = 1$ to model a uniform flow whereas, following Euler, the "uniform flow" is appropriately expressed by $\Delta\tau = 0$, or according to Eq. (5.7)

$$\Delta.\ t = \tau \qquad \Delta\Delta.\ t = \Delta\tau = 0. \tag{5.14}$$

In contrast to the algorithm Newton had introduced in the *Method of Fluxios*, in Euler's calculus of differences another algorithm is invented which allows the theoretician to establish a *one-to-one relation* between variables and increments. Variables and increments are always quantities of the "same sort",[45] i.e. if the variable is not composed of other quantities, the increment cannot be composed of other quantities, too. Following Euler, the *increment* ω is always not only treated as an *indivisible* quantity, but also as a quantity which is not composed of other quantities (see Fig. 5.1). The indivisibility of the increment is not postulated, but algebraically represented by the relation between the variable and the increment, i.e. $\Delta.\ x = \omega$ which had been defined in the calculus of finite differences. The increment of a constant quantity is zero $\Delta.\ a = 0$. Hence, in the calculus of finite differences, the increment of a variable is indeterminate whereas the increment of a constant quantity, although being zero, is determinate.[46]

The variables and their increments investigated in mechanics are confined to spaces $s, \Delta s$, times $t, \Delta t$ and velocities $v, \Delta v$ where the velocity is defined by the

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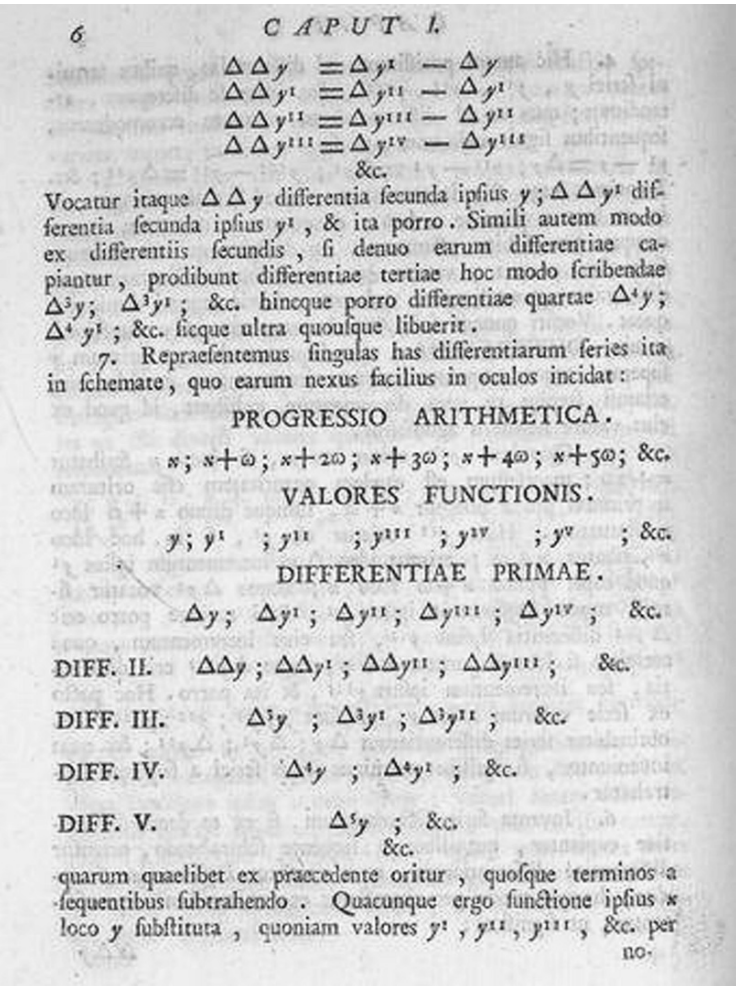
[45] Hence, modifying Euler's statement about space and time, i.e. "18. The ideas of space and of time have almost always been of the same sort, so that those who have denied the reality of the one have also denied of the other, and conversely" [Euler E149, § 18], we may assert, that "variables and their increments have always to be of the same sort".

[46] "It is evident that the differential calculus is a special case of the calculus which I have enunciated above, since what was before assumed arbitrary is now taken to be infinitely small." [Euler 1727] Following Euler, although the increment is of infinitesimal magnitude it is determinate with respect to any finite quantity by Bernoulli's rule, i.e. it is only taken to be zero in case of arithmetical ratios, (i) $a + n\ dx - a = 0$, but it is taken different from zero in case of geometrical ratios, (ii) dy/dx, but, by Eq. (i) it is less than any finite number since the magnitude of the finite number is not modified, i.e. it is neither increased not decreased by its addition to or subtraction of a "cipher" like *ndx* [Euler E212, §§ 84–86]. The theorem on the magnitude, i.e. "less than any finite number, but different from zero" is formulated for the geometrical model of a mass point [Euler E015/016, § 134]. "Quomodo autem se habeant diversam potentiarum effectus, mox sumus expposituri, atque etiam in punctis, quae a potentiis sollicitantur, diversitatem ponemus, ut aliud in data ratione maius minusve esse possit. Neque vero haec inaequalitas adversatur extremae punctorum parvitati, non enim puncta mathematica intelligimus, sed physica, ex quorum compositione corpora oriuntur. Possunt enim duo plurave in unum coalescere concipi, quod, quanquam simplicibus est maius, infinite tamen exiguae manet magnitudinis." [Euler E015/016, § 134]



6 CAPUT I.

$$\Delta\Delta y = \Delta y^{I} - \Delta y$$
$$\Delta\Delta y^{I} = \Delta y^{II} - \Delta y^{I}$$
$$\Delta\Delta y^{II} = \Delta y^{III} - \Delta y^{II}$$
$$\Delta\Delta y^{III} = \Delta y^{IV} - \Delta y^{III}$$
&c.

Vocatur itaque $\Delta\Delta y$ differentia ſecunda ipſius y; $\Delta\Delta y^{I}$ differentia ſecunda ipſius y^{I}, & ita porro. Simili autem modo ex differentiis ſecundis, ſi denuo earum differentiae capiantur, prodibunt differentiae tertiae hoc modo ſcribendae $\Delta^3 y$; $\Delta^3 y^{I}$; &c. hincque porro differentiae quartae $\Delta^4 y$; $\Delta^4 y^{I}$; &c. ſicque ultra quouſque libuerit.

7. Repraeſentemus ſingulas has differentiarum ſeries ita in ſchemate, quo earum nexus facilius in oculos incidat:

PROGRESSIO ARITHMETICA.

x; $x+\omega$; $x+2\omega$; $x+3\omega$; $x+4\omega$; $x+5\omega$; &c.

VALORES FUNCTIONIS.

y; y^{I}; y^{II}; y^{III}; y^{IV}; y^{V}; &c.

DIFFERENTIAE PRIMAE.

Δy; Δy^{I}; Δy^{II}; Δy^{III}; Δy^{IV}; &c.

DIFF. II. $\Delta\Delta y$; $\Delta\Delta y^{I}$; $\Delta\Delta y^{II}$; $\Delta\Delta y^{III}$; &c.

DIFF. III. $\Delta^3 y$; $\Delta^3 y^{I}$; $\Delta^3 y^{II}$; &c.

DIFF. IV. $\Delta^4 y$; $\Delta^4 y^{I}$; &c.

DIFF. V. $\Delta^5 y$; &c.

&c.

quarum quaelibet ex praecedente oritur, quoſque terminos a ſequentibus ſubtrahendo. Quacunque ergo functione ipſius x loco y ſubſtituta, quoniam valores y^{I}, y^{II}, y^{III}, &c. per

no-

Fig. 5.1 Euler's calculus of finite differences [Euler E212, §§ 6 and 7]

relation $\Delta s = v \cdot \Delta t$. These mechanical quantities had been completed by mass and forces (compare Chap. 2). The requirements of the calculus of finite differences are completed by the requirements of *mechanical homogeneity*. An increment of a line is also a line of finite length, an increment of any duration is also a duration represented by a finite time interval. It is said that the quantities are of different dimension.[47] Hence, a relation $\Delta s \sim \Delta t$ between two increments of *different* di-

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47 *Constants, units, measures and dimensions in Leonhard Euler's mechanics*, Francisco A. Gonzáles Redondo, Kendrick Press [Redondo]. Euler's investigations of arithematical and mechanical homogeneity and the use of the notion "dimension" are analyzed. See also *Nova methodus innumerabiles aequationes differentiales secundi gradus reducendi ad aequationes differentiales primi gradus* presented 1728, published 1732 [Euler E010].

mension is mathematically indeterminate as long as it is not assumed that either the path is a function of time or the time is a function of path, i.e. either $s = s(t)$ with $\Delta t = \text{const}$ or $t = t(s)$ with $\Delta s = \text{const}$. The experimental condition has to be adapted to the mathematical representation of the *mechanical* quantities in the calculus of increments keeping either the time intervals or the spatial intervals in the arithmetical progressions $\alpha,\ \alpha+\omega,\ \alpha+2\cdot\omega,\ \alpha+3\cdot\omega,\dots$ with $\alpha = s,t$ and the corresponding increments $\omega = \Delta s, \Delta t$ invariant by asserting either $\Delta t_{\text{exp}} = \Delta t = \text{const}$ or $\Delta t_{\text{exp}} = \Delta t = \text{const}$, respectively. It is not necessary to express the increment of a distance by an increment of a time interval if the measurements are performed independently of each other.

All these relations between mechanical quantities and all these arguments to establish mathematical and mechanical homogeneity remain to be valid if the increments become *infinitesimal* with the only lack in comparison to any finite quantity, i.e. the lacking of experimental confirmation. Neither the mathematical nor the mechanical homogeneity depends on a certain numerical value of any or some of the quantities which constitute the basic mechanical relations even if they had been formulated as axioms.[48]

Following Leibniz, the "minimal motion" is less than any assignable motion, but different from zero, therefore, it is called "evanescent motion" with respect to any finite motion, but not in relation to the "absolute zero", i.e. nothing. However, this quantity called "absolute zero" is indispensable for reckoning to *truncate* all terms which disturb the representation of the result in terms of "*non-evanescent* quantities", i.e. finite and assignable quantities. The usually obtained result is given in terms of a *mixed representation*, i.e. it is made up (i) of *finite* quantities $x,y,z,\dots$ and *finite* increments $\Delta x, \Delta y, \Delta z, \dots, \Delta\Delta x, \Delta\Delta y, \Delta\Delta z, \dots$ or (ii) of *finite* quantities $x,y,z,\dots$ and *infinitesimal* increments or differentials $dx, dy, dz, \dots ddx, ddy, ddz, \dots$.

In the calculus of finite differences the *increment* Δf^I of the function depends on two independent variables, first (i) on the independent variable x and, second (ii) on the magnitude of the increment ω, i.e. for any chosen value of the variable x, the increment of the function becomes a pure function of the increment. However, examining the general formula for the increment of the function

$$\Delta f^I(x,\omega) = P(x)\omega + Q(x)\omega^2 + R(x)\omega^3 + \cdots, \tag{5.15}$$

it follows that it is possible to construct an *increment independent* term by calculating the ratio of the increment of the function and the increment of the independent variable.

$$\frac{\Delta f^I(x,\omega)}{\omega} = P(x) + Q(x)\omega + R(x)\omega^2 + \cdots, \tag{5.16}$$

---

[48] Newton gave the basic law not as an equation, but in term of "being proportional". "The change of motion is *proportional* to the impressed moving force (...).", "Mutationem motus *proportionalem* esse vi motrici impressae, (...)." [Newton, Principia, Axioms]

The same procedure can be applied to differences of any order.

$$\frac{\Delta\Delta f^I(x,\omega)}{\omega^2} = Q(x) + R(x)\omega + \cdots, \tag{5.17}$$

$$\frac{\Delta^3 f^I(x,\omega)}{\omega^3} = R(x) + S(x)\omega + \cdots. \tag{5.18}$$

In the calculus of finite differences, the order of the difference and the powers of the increments are not uniquely related to each other. Although the degree of the difference is always related by a one-to-one correspondence to the *lowest* power of the increment, i.e. the n-th difference cannot depend on the (n−1) power of the increment, any of the differences can additionally depend on increments of higher power. Following Leibniz, it can be assumed that, in case of differentials, these higher order terms of differentials are less than the leading term. Consequently, all higher order terms can be “neglected”, i.e. the power series is truncated after the first term independently of the degree of the differences.[49]

Following Euler, the differential calculus is obtained from the calculus of finite increments without any modification of the algebraic form of the algorithms which had been derived for the increment of a function in dependence on the increment of the variable.[50] The finite increment of a function becomes infinitesimal if the finite increment of the variable is substituted by an infinitesimal increment. The algebraic structure of the analytical expression is preserved.

$$\Delta f^I = P\Delta a + Q\Delta a^2 + R\Delta a^3 + \cdots, \tag{5.19}$$

$$df^I = Pda + Qda^2 + Rda^3 + \cdots. \tag{5.20}$$

In the following step, the magnitude of the terms which appear in (5.19) and (5.20) is investigated. Following Euler, the numerical relations are only completely determinate for Eq. (5.20) since the term Qda^2 is less than any assignable quantity in comparison to the term Pda and, by the same reasons the term Rda^3 is less than any assignable quantity in comparison to Qda^2 [Euler E212, § 88].

> Hence, if we follow the usage of exponents, we call dx infinitely small of first order, dx^2 of the second order, dx^3 of the third order, and so forth. It is clear that in comparison with an infinitely small quantity of the first order, those of higher order will vanish. [Euler E212, § 88]

[49] Here, it is crucial to distinguish between those terms which are really *differences* being related to functions and those which are not differences, but *increments of variables*, i.e. *df(x)* and *dx*, respectively. *The relations between the increments are independent of the relations between the increments and the variables*. Hence, the relations between increments of different power are also independent of the relation between variable and increments. Mechanically, the independence of the increment of the basic variable is represented by the equation of motion, i.e. the change in velocity is independent of the velocity [Euler E015/016, § 131].

[50] Compare Leibniz on homogeneity. The paper is entitled “Symbolismus memorabilis calculi algebraici et infinitesimalis in comparatione potentiarum et differentiarum et de lege homogeneorum transcendentali”. “(...) sed quia *dx* vel *dy* est incomparabiliter minus quam *x* vel *y*, etiam *dx dy* erit incomparabiliter minor quam *xdy* est *ydx*, ideoque rejicitur, (...).” [Leibniz, Homogeneity]

Then, the only remaining term is the first term of the series and the increment of function is completely determinate.

$$df^I = Pda \tag{5.21}$$

The same procedure is applied to the increments of the function of higher order.

$$\Delta\Delta f^I = Q\Delta a^2 + R\Delta a^3 + \ldots, \qquad ddf = Qda^2 \tag{5.22}$$

$$\Delta^{(3)} f^I = R\Delta a^3 + \ldots, \qquad d^{(3)} f = Rda^3 \tag{5.23}$$

Every order n of the increment of the function is automatically related the term depending on the n-th power of the increment Δa^n and higher order contributions, but never terms which are of an order less than n. Lower order terms are already automatically excluded by the algorithms of the calculus of *finite* differences. Hence, it is impossible that they are recovered by an only change of the magnitude of increments and one can take advantage from the distinction between the algebraic structure of the expressions and the magnitude of the increments. The algebraic structure is *independent* of the magnitude of increments, but is solely determinate by the different increments of the function $\Delta f,\ \Delta\Delta f,\ \Delta^{(3)} f$&c. whose existence depends also only on the type of function, e.g. in case $f = x^2$ it follows $\Delta f \neq 0,\ \Delta\Delta f \neq 0$, whereas $\Delta^{(3)} f = 0$ and all higher order differences are equal to zero (compare Eq. (5.10)).

The direct relation between the calculus of differences and the differential calculus was preserved for a long time.[51] It was also important for the representation of mechanics independent of geometrical methods [Euler E015/016].[52] Already in 1727, Euler commented on the relevance of the method for the foundation of the differential calculus:

> If radical[53] quantities increase by finite quantities, the calculus might be called the *calculus of finite differences* or *finite increments*. It should not be confounded with the calculus in which radical quantities can assume infinitely small increments. This case will be discussed later on, after the exposition of the calculus of finite differences. To the best of my knowledge nobody yet has developed the latter, so that it might seem unnecessary to use it for the study of the rules of differential calculus. However, when I saw that quite a lot scholars including those not unexperienced in this higher analysis, do not have any correct, or even frequently possess a false, idea about the calculus of differences I thought it inappropriate to

[51] In 1804, Lacroix reviewed the development of the *calculus of differences*. "CHAP. I. DU Calcul des Différences, Methodus differentialis (Newtoni opuscula). Methodus incrementorum (Taylor). Philosophical transactions (n° 353, ann. 1717, p. 676). Mém. Acad. des Sciences de Paris, années 1717, 1723, 1724 (Nicole). Methodus differentialis, sîve Tractatus de summatione et interpolitione serierum (Stirling). Essays on Several curious and useful subjects, p. 87 (Th. Simpson). Institutiones Calcul. diff. Pars I, cap. I et II (Euler). The Method of increments (Emerson). Théorie générale des équations, introduction (Bézout). Methode directe et inverse des différences, ou leçons d'Analyse données à l'École." [Lacroix, Vol. 3]

[52] Although Euler presented the extended version of the foundation only later in 1755 [Euler E212], he made use of the complete and consistent representation for the invention of analytical mechanics [Euler E015/016].

[53] Jushkevich commented: "Quantities forming the function are called 'radical' (quantitates radicales'). It seems that no other scholar used this term in Euler's meaning." [Jushkevich, Euler]

omit this calculus and decided to explicate it before going over to the differential calculus. [Euler 1727]

Following Johann Bernoulli, Euler developed a complete and consistent algebraic and arithmetical treatment of the calculus of finite differences and transferred the rules developed and demonstrated for *finite* increments to the reckoning procedures with *infinitesimal* increments. In the true interpretation and by its nature in goal and spirit, the treatise on the *Calculus of differentials* written by Euler in 1727 is a *Calculus of increments* being related to the Newtonian and Leibnizian calculus by the same principles which had been invented by Taylor in 1715 in the treatise entitled *Methodus Incrementorium directa & inverse* [Taylor, Methodus]. Later in 1895, Felix Klein acknowledge the approach which had been invented by Taylor and Euler.[54]

5.2.2 Infinitesimal, Finite and Infinite Quantities

Following Leibniz, the status of differentials is to be determinate in relation to quantities of finite magnitude. The differentials are less than any assignable quantity. Therefore, Leibniz concluded that differentials are fictititous quantities (compare Chap. 3). Nevertheless, acknowledging the merits of Newton and Leibniz in the invention of the calculus and notwithstanding the different interpretations of both authors and the criticism by other authors, Euler claimed that Leibniz brought the calculus into the shape of a science or a system of rules [Euler E212, Preface LXIII].[55] This system of rules was reinterpreted and extended by Euler who accentuated the common properties finite quantities and differentials and included infinite quantities into the consideration. Following Euler, the common properties of all different kinds of quantities consists in the possiblity of augmentation or diminution without limit.

[54] "Ich möchte zuerst darauf hinweisen, daß das von Taylor zwischen Differenzen- und Differentialrechnung geknüpfte Band noch lange Zeit gehalten hat: Noch in den analytischen Entwicklungen Eulers gehen beide Disziplinen Hand in Hand, und die Formalen der Differentialrechnung erscheinen als Grenzfälle ganz elementarer Beziehungen, die in der Differenzenrechnung statthaben. Diese so naturgemäße Verbindung wurde erst durch die wiederholt erwähnten formalen Definitionen des Lagrangeschen *Derivationskalküls* aufgehoben. (...) das ganz auf Lagrangeschem Boden alle damals bekannten Tatsachen der Infinitesimalrechnung zusammenfaßt, den *Traité du calcul différentiel et du intégral* von *Lacroix* (...). So ist diese Formel auf eine vollkommen veräußerlichte, allerdings nicht angreifbare Weise hervorgebracht. In diesen Gedankenkreisen konnte Lacroix natürlich die Differenzenrechnung als Ausgangspunkt nicht mehr benutzen; sie erscheint ihm aber doch für die Praxis zu wichtig, als daß er sie weglassen wollte, und so ergreift er denn den Ausweg, sie in ganz selbständiger, übrigens ausführlicher Darstellung hinterher im dritten Bande zu bringen, ohne daß gedankliche Brücken von ihr zur Differentialrechnung führen." [Klein, Elementarmathematik, p. 253]

[55] "LEIBNIZIO autem non minus sumus obstricti, quod hunc calculum, ante hac tantum velut singulare artificium spectatum, in formam discipline redegerit, eiusque praecepta tanquam in systema collegerit, ac dilucide explicaverit." [Euler E212, Preface LXIII]

> 1. Whatever is capable of increase or diminution, is called *magnitude*, or *quantity*. (...) 2. Mathematics, in general, is the *science of quantity*; or; the science which investigates the means of measuring quantity. [Euler E387, §§ 1 and 2]

Although finite, infinitesimal and infinite quantities are clearly distinguished from each other by their essentially different magnitudes, these quantities of different type are, nevertheless, quantities and can be increased or decreased according to general properties any quantity should obey.[56] Furthermore, Euler generalized the methology which had been introduced by Leibniz to establish common principles in *geometry* and *mechanics* by the principle of continuity [Leibniz, Specimen, II (4)] (compare Chaps. 2 and 3). Leibniz called this criterion (touchstone) principle of continuity because the main purpose was to exclude jumps or leaps from nature and thinking. This consequences had been discussed for the continuity of motion and, as far as methology is concerned, for the transfer of the given order of suppositions into the order of consequences or conclusions [Leibniz, Specimen, II (4)]. Following Euler, the calculus of differences is not only a powerful *algorithm* for reckoning, but additionally provides support as an *ordering* scheme in mathematics and mechanics. In mechanics, the order is established by the one-to-one correspondence between mathematically represented differences of 1st and 2nd order and the corresponding mechanical quantities velocity and force (acceleration): (i) the velocity is related to the spatial intervals $\Delta s = s^I - s$, i.e. the difference of positions, by the relation $\Delta s = v \cdot \Delta t$ and (ii) the force (acceleration) is related to the spatial intervals $\Delta\Delta s = s^{II} - 2s^I + s$, i.e. the difference of the difference of positions, by the relation $\Delta\Delta s = (K/m) \cdot \Delta t^2$, respectively. As it had been demonstrated in Sect. 5.2.1, this order is preserved for infinitesimal quantities or the *differentials* and *differentio – differentials*, i.e. $ds = v \cdot dt$ and $dds = (K/m) \cdot dt^2$, respectively.

Referring to Leibniz, Euler conclude that a transition from finite to infinite and and from finite to infinitesimal quantities is possible if the principle of continuity is transferred to series of discrete numbers. Euler claimed that

> 98. Quantitates tum infinite parvae, quam infinite magna in seriebus numerorum saepissime occurunt, in quibus cum sint numeris finitis permixtae, ex iis luculenter patebit, quemadmodum secundum leges continuitatis a quantitatibus finitis ad infinite magnas atque infinite parvas transitio fiat. Consideremus primum serium numerorum naturalium, quae simul retro continuata erit
>
> $$\&c. -4, -3, -2, -1,\ 0, +1, +2, +3, +4, +\&c.$$
>
> Numeri ergo continuo decrescendo praebent tandem 0 seu infinite parvum, unde ulterius continuitati negativi evadunt. Quamobrem hinc intelligitur a numeris finitis affirmativis decrescentibus transiri per 0 ad negativos crescentes. [Euler E212, § 98][57]

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[56] Here, Euler made use of the same procedure he had successfully developed for the definition of the general properties of bodies [Euler E842, Chap. 1] (compare Chap. 4). "4. Was allen Körpern ohne einige Ausnahmen zukommt, wird eine Eigenschaft der Körper genannt, und daher werden alle Dinge, in welchen sich diese Eigenschaft nicht findet, von dem Geschlecht der Körper ausgeschlossen." [Euler E842, § 4]

[57] "98. From this it follows with full clarity, how, according to the laws of continuity, one passes from finite quantities to infinitely small and to infinitely large quantities." [Euler E212, § 98]

The *finite* increment of path $\Delta s = v \cdot \Delta t + b \cdot \Delta t^2 + \cdots$ can be expressed in terms of *finite* increments of time Δt, but never in terms of infinitesimal increments of time dt. By the same reason, the *infinitesimal* increment of path $ds = v \cdot dt$ can be expressed in terms of *infinitesimal* increments of time dt, but never in terms of finite increments of time Δt. These relations are governed by the theorem that "infinitesimal quantities are less than any assignable quantity" where "assignable" means "finite" quantity. In terms of measurement, any finite quantity is measured by comparison to a finite unit. The result of the measurement, i.e. the measurement of the length of a body L, is represented by a *finite* number whose magnitude tells us how many times the unit of length l_{unit} is contained in the length of the measured object, $L/l_{\text{unit}} = n$.

> 3. Now, we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known piece of money, a crown, a ducat, or some other coin, and shew how many of these pieces are contained in the given sum. In the same manner, if it were proposed to determine the quantity of a weight, we should take a certain known weight, for example, a pound, an ounce, &c. and then shew how many times one of these weights is contained in that which we are endeavouring to ascertain. [Euler E387 (London), § 3]

Obviously, such a relation cannot be established for the measuring of a finite length if the "unit" is represented by an infinitesimal quantity $d\lambda$. Nevertheless, the same procedure can be applied for the "measurement" of the infinitesimal increment of the path ds in terms of $d\lambda$, i.e. $ds/d\lambda = m$, where the result is also a finite number. The magnitude of m depends on te magitude of ds, for $ds_1 < ds_2$ it follows $m_1 < m_2$ since the unit $d\lambda = \text{const}$ is assumed to be invariant or independent of the measurement. Accordingly, the increment of the independent variable, e.g. of time, is also assumed to be a constant quantity[58] $dt = c = \text{const}$ or $d(dt) = dc = 0$.

In the *arithmetical* ratio of a finite and an infinitesimal quantity represented by $x \pm n \cdot dx$, none of these quantities is a measure of the other quantity. In the *geometric* ratio either of two finite quantities a and b or of two infinitesimal quantities dy and dx, however, one of the quantities can be regarded as the measure or the unit with respect to the other. Neither a nor b is unassignable less or unassignable large with respect to the other one. The same statement holds for the infinitesimal quantities dy and dx since they are of the same order of smallness. Hence, Euler stated[59]:

> 86. Simili modo, si diversa occurunt infinite parva *dx* & *dy*, etiamsi utrumque sit = 0, tamen eorum ratio non constat. Atque in investigatione rationis inter duo quaeque huiusmodi infinite parva omnis vis calculi differentialis versatur. [Euler E212, § 86][60]

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[58] Independently of geometry by analytical arguments, the rule $da = 0$ for $a = \text{const}$ had been invented by Leibniz [Leibniz, Elementa] in 1680 and published for the first time in 1684 [Leibniz, Nova methodus] (compare Chap. 3).

[59] Following Euler, the whole force of the calculus is included in the study of the geometrical ratio of differentials *dx* & *dy*. [Euler E212, § 86]

[60] "86. For this reason these two infinitely small quantities *dx* and *a dx*, both being equal to 0, cannot be confused, when we consider their ratio. In a similar way, we will deal with two infinitely small quantities *dx* and *dy*. Although these are both equal to 0, their ratio is still unknown. Indeed the whole force of differential calculus is concerned with the investigation of ratios of

For this reason, Euler recommended to be cautious in the application of algorithms since the infinitesimal quantities are different from each other and, consequently, had to be denoted by different signs.[61] Otherwise, one "may fall into the greatest confusion with no way to extricate ourselves." [Euler E212, § 85][62]

> 85. Haec autem etiam in vulgari Arithematica sunt planissima: cuilibet enim notum est, cyphram per quemvis numerum multiplicatam dare cyphram, esseque n.0 = 0 [n.cyph1 = cyph2], sicque fore n : 1 = 0 : 0 [n : 1 = cyph2 : cyph1]. Unde patet fieri posse, ut duae cyphrae quamcumque inter se rationem geometricam teneant, etiamsi, rem arithmetice spectando, earum ratio semper fit aequalis. Cum igitur inter cyphras ratio quaecumque intercedere possit, ad hanc diversitatem indicandam consulto varii characteres usurpantur; praesertim tum, cum ratio geometrica, quam cyphrae variae inter se tenent, est investiganda. In calculo autem infinite parvorum nil aliud agitur, nisi ut ratio geometrica inter varia infinite parva indagetur, quod negotium propterea, nisi diversis signis ad ea indicanda uteremur, in maxime confisuionem illaberetur, neque ullo modo expediri posset. [Euler E212, § 85]

The essential difference between arithmetical and geometrical ratio of differentials can be readily demonstrated in mechanics. Although $dt = \text{const}$, the differentials and differentio-differentials can be increased or decreased in magnitude by a change of velocity or force or mass, i.e. $ds = v \cdot dt$ and $dds = (K/m) \cdot dt^2$, respectively, because these finite quantities can be experimentally modified. In case the interaction of two bodies, the masses can be chosen differently, $m_1 < m_2$, then, there are different changes of velocity $dv_1 \neq dv_2$ related to each other by the equation $m_1 dv_1 + m_2 dv_2 = 0$ (compare Chap. 4). Once more, the geometric ratio of infinitesimal quantities can be expressed in terms of a finite quantity, the inverse ratio of masses, $dv_1/dv_2 = -m_2/m_1$.

In 1755, Euler made use of all mathematical and logical tools he had developed earlier for the investigation of other mathematical, mechanical and method-

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any two infinitely small quantities of this kind." [Euler E212, § 86] Michelsen translated: "In der Infinitesimal=Rechnung aber thut man nichts anders, als daß man sich mit der Untersuchung des geometrischen Verhältnisses zwischen verschiedenen unendlich kleinen Größen beschäftigt, und dabey würde man in die größte Verwirrung gerathen [aus der man sich nicht befreien könnte], wofern man nicht diese unendlichkleinen Größen mit verschiedenen Zeichen bezeichnete." [Euler E212, § 85 (Michelsen)]

[61] Later, Lagrange interpreted Euler's approach just in the sense Euler intended to exclude. Lagrange claimed that an expression "zero divided by zero" or, analytically 0/0, gives no idea. However, this is just that interpretation Euler warned against and intended to prevent the reader. "On connaît les difficultés qu'offre la supposition des infiniment petits, sur laquelle *Leibnitz* a fondé le calculs différentiel. Pour les éviter, Euler regarde les différentielles comme nulles, ce qui réduit leur rapport à l'expression *zéro* divisé par *zéro*, laquelle ne présente aucune idée." [Lagrange, Fonctions, Lecon Première]

[62] "85. (...) Since between ciphers (zeros) any ratio is possible, in order to indicate this diversity we use different characters (notations) on purpose, especially when a geometric ratio between two different ciphers (zeros) is being inverstigated. In the calculus of the infinitely small nothing else is carried out as to investigate the geometric ratio between different infinitely small, where we fall, therefore, unless we have indicated them with different signs, into the greatest confusion with no way to extricate ourselves." [Euler E212, § 85]. However, the representation of the ratio $n : 1 = 0 : 0$ (instead of $n : 1 = \text{cyph1} : \text{cyph2}$) does not comply with this rule.

ological problems. Especially,[63] Euler's procedure may be evidenced in its application in Chap. III entitled *On the infinite and the infinitely small* (De infinitis et infinite parvis). Euler made use of the papers written before on *Calculis differentialis* [Euler 1727], *Gedancken von den Elementen der Cörper* [E081] and *Anleitung zur Naturlehre* [E842] written in 1727, in 1746 and after 1746, respectively. The advantageous application of logical argument by Euler should be acknowledged. Euler split his criticism into (i) a *purely logically* based parts where no mechanical or mathematical principles are analyzed, but the main topic is the analysis of the use the authors made of their frame of reference (which had been invented before) and (ii) the conclusions related to the validity of the mechanical concepts which are based on the models taking into account the results of the previous logical analysis.[64]

5.2.3 Topological Interpretation

Following Euler, the motion of a body which is driven by forces can be described by a reference system made up of two geometric objects, first of a plane and, second of a straight line oriented perpendicular to the plane. The intersection between plane and straight line is uniquely defined. Then, it is assumed that a body of infinitesimal magnitude is positioned at this intersection point (see Fig. 5.2). The point is distinguished from all other regions of the plane as well the straight line by the *assignment* of a mass of *finite* magnitude m. Euler assumed that the position of the body is changed in the presence of a force whose direction is oriented perpendicular to the plane.[65] According to Newton's 2nd Law, the translation of the body out of the plane depends on the direction of the force, either $+K$ or $-K$. [Newton, Principia, Axioms]. Independent of the direction, the body is pushed *outside* the plane in both cases. Hence, the position of the body can be *topologically* determinate with respects its *position* (*situs*)[66] to the plane described independently of the magnitude of translation. Once

[63] "I De differentiis finitis. II. De usu differentiarum in doctrina serierum. III. De infinitis et infinite parvis. IV. De differentialium cuiusque ordinis natura. V. De differentiatione functionum algebraicarum unicam variabilem involventium. VI. De differentiatione functionum transcendentium. VII. De differentiatione functionum duas pluresve variabiles involventium. VIII. De formularum differentialium ulteriori differentiatione. IX. De aequationibus differentialibus." [Euler E212]

[64] "16. Sollte man aber bei fleissiger Untersuchung diese Gründe nicht nur unrichtig befinden, sondern auch nach Verbesserung derselben, durch rechtmässige Schlüsse auf eine ganz andere Lehre von den Elementen der Körper gerathen: so würde dadurch nicht nur die Unrichtigkeit des Leibnizschen Lehrgebäudes von den Monaden deutlich dargethan, sondern auch an desselben Stelle die wahre Beschaffenheit der körperlichen Dinge erkannt werden." [Euler E081, II, § 16]

[65] "20. (...) Pour déterminer le mouvement de ce corps, on n'a qu'à avoir égard à l'éloignement de ce corps d'un plan quelconque fixe et immobile; soit à l'instant présent la distance du corps à ce plan $= x$; qu'on décompose toutes les forces qui agissent sur le corps, selon directions qui soient ou parallèles au plan, ou perpendiculaires et soit P la force qui résulte de cette composition selon la direction perpendiculaire au plan et qui tachera par conséquent ou à éloigner ou à rapprocher le corps du plan (...)." [Euler E177, § 20] (Compare also [Euler E842, § 69])

[66] According to Leibniz's *analysis situs* [Leibniz, Initia].

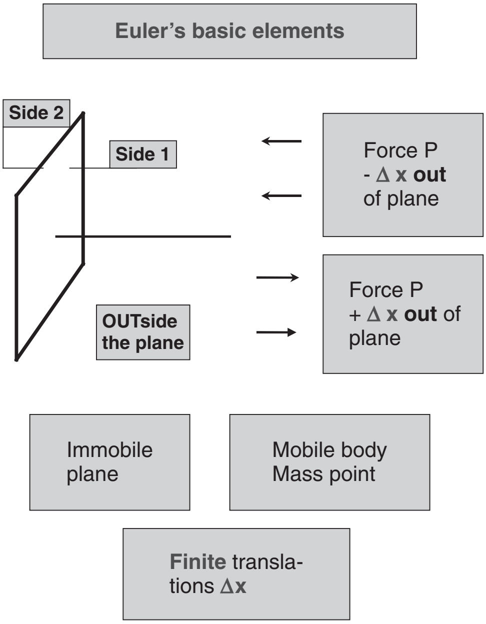

Fig. 5.2 Euler's topological model for the relation between planes, lines, translations and forces. *Finite* forces and *finite* translations
Note: The force P pushes the body away from the plane into the region outside the plane, either outside of the side 1 or outside of side 2. The is no motion inside the plane, i.e. $\Delta y = \Delta z = 0$. The transaltion is assumed to be of finite magnitude $\pm\Delta x$.

being established, the topological relations are not modified having specified the magnitude of the translation in dependence on the magnitude of the force. The force P pushes the body away from the plane into the region outside the plane, either outside of the side 1 or outside of side 2. There is no motion *inside* the plane, i.e. $\Delta y = \Delta z = 0$, since there is no component of the force inside the plane The translation is assumed to be of finite magnitude $\pm\Delta x$. The *topological* relations between *forces* and *translation* of the body out of the plane are also preserved for any magnitude of the translation, i.e. the translation is independent of the magnitude of forces provided that the condition $P \neq 0$ is fulfilled. Hence, the finite translation can be replaced with infinitesimal translations $\pm dx$ pushing the infinitesimal body outside the plan whereas inside the plane there is no translation at all, i.e. as before described by the relation $dy = dz = 0$. Hence, the topologically determinate difference results in the relation $\pm dx \neq 0$ or, $-dx < 0 < +dx$. The magnitude of the translation out of the plane depends on the product of the force and the square of the time element $\Delta\Delta x \sim K\Delta t^2$ where Δt is the time the force is acting upon the body. In case of infinitesimal time element (tempusculum) it follows $mddx = Kdt^2$ [Euler E015/016, § 152]. The force prevents the body to stay at his place (position) in the plane. The topological invariance ensures the validity of the mechanical relations in

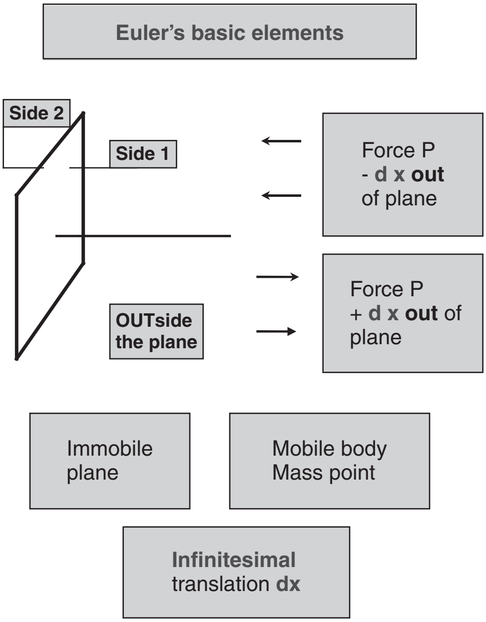

Fig. 5.3 Euler's topological model for the relation between planes, lines, translations and forces. *Finite* forces and *infinitesimal* translations
The force P pushes the body away from the plane into the region outside the plane, either outside of the side 1 or outside of side 2. The is no motion inside the plane, i.e. $dx = dy = 0$. Translation is assumed to be of infinitesimal magnitude $\pm dx$.

boths cases where the relations between time, space, mass and forces are represented either only in terms of finite or in terms of finite and infinitesimal quantities.

5.3 Algorithms

In the Preface of the treatise on the calculus entitled *Institutiones calculi differentialis* [Euler E212] Euler acknowledged the merits of Leibniz in the invention of the calculus as a science. Before Leibniz, Euler claimed, the calculus had been regarded rather as a kind of art than a science. In Chap. 4 of the *Institutiones* [Euler E212] Euler systematically developed all algorithms necessary for reckoning and the formulation and solution of mathematical problems related to the calculus. Euler exclusively based the complete set of algorithms for the *differential calculus* on the *calculus of differences*. Thus, in contrast to Newton and Leibniz who implicitly invented some essential algorithms, Euler always made use of all explicitly demonstrated rules known from the calculus of finite increments. Then, the transfer to the differential calculus is straightforwardly since the differential calculus is assumed to be a special case of the calculus of differences [Euler E212, §§ 112–114, 118–121] (see Fig. 5.3).

In Chap. III Euler systematically developed all algorithm necessary for reckoning and the formulation and solution of mathematical problems. Following Leibniz and Newton, Euler invented a complete set of algorithms for the differential calculus based on the calculus of differences or finite increments. In contrast to Newton and Leibniz who implicitly invented some essential algorithms, Euler exclusively made use of all explicitly demonstarted rules obtained for finite increments. Then, the transfer to the differential calculus is starightforwardly since the differential calculus is assumed to be a special case of the calculus of differences [Euler E212, §§ 112–114, 118–121].

> 114. Erit ergo Analysis infinitorum, quam hic tractare coepimus, nil aliud, nisi casus particularis methodi differentiarum in capite primo expositae, qui oritur, dum differentiae, quae ante finitae erat assumtae, statuantur infinite parvae. [Euler E212, § 114]

Furthermore, Euler carefully distinguished between (i) the algebraic part defined for the calculus of differences and the (ii) establishing (constituting) of the special case. The latter procedure includes an extension of the *numerical basis* which was usually formed by the integers, the rational and the irrational numbers. All these number had been regarded to be of finite magnitude with the only exception of the number zero. Euler distinguished between infinitesimal, finite and infinite quantities. Following Euler, all analytical expressions are to be classified by "before" and "after" it is assumed that the infinitesimal quantities are nothing else as zeros [Euler 1727], [Euler E212, § 83]. In the state "before", the analytical expressions are to be "completely prepared" [Euler 1727].[67]

The set of algorithms is as complete as possible and its different parts are internally perfectly interconnected that it can only be completely accepted or completely rejected. D'Alembert and Lagrange proceeded differently. D'Alembert introduced the concept of limits [d'Alembert, Encyclopédie] whereas Lagrange rejected both the ideas of limit and the infinitesimal quantities [Lagrange, Fonctions]. The transition from the calculus of differences to the calculus of differences had been mathematically formulated as a selection problem since, instead of increments being variables or indeterminate quantities $\Delta x, \Delta y, \Delta z, \ldots$, now a *determinate* magnitude of increments is assumed represented b the differentials $dx, dy, dz, \ldots$ [Euler E101, §§ 1–6]. In contemporary terminology, Euler's assumption is represented by the infinitesimal quantities are labeled by $\Delta_{\text{inf}} x \approx 0$, $\Delta_{\text{inf}} y \approx 0$ and $\Delta_{\text{inf}} z \approx 0$, [Keisler].

Euler is completely right in stating that, in certain respect, the handling of the differential calculus is easier than the treatment of any version of the calculus not finite differences [Euler E212, § 121].[68] Following Euler's procedure, the general result is that an infinite series is replaced with a polynomial. This transition can also be interpreted in terms of a selection problem of one of functions $P(x), Q(x), R(x), \ldots$

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[67] "At the same time Euler warned against errors which may happen when the differentials or infinitesimals are omitted before the given expression 'is completely prepared' ('*quam expressio penitus est adornata*')." [Euler 1727, (Juskevich)]

[68] "121. Differentials are much easirer to find than finite differences. For the finite difference (...) it is not sufficient to know *P*, but we must investigate also the functions *Q, R, S*, etc." "121. Differentilia igitur multo facilius inveniuntur, quam differentiae finitae." [Euler E212, § 121]

the infinite series is made up. Following Euler [Euler E212, §§ 90–93, 112–114], the function $P(x)$ should be necessarily finite if the first difference or the differentials is expressed in terms of the first differential or the infinitesimal increment of the variable x. Hence, from the application of Euler's theory a criterion is derived for the choice of functions being possible candidates to describe a mechanical system (compare Chap. 8).

The algorithm are presented at first analytically in terms of (i) variables, constants and ciphers and numerically in terms (ii) of variables, constant and zeros and, at second in terms of (iii) functions depending on variables and constants and (iv) on increments and decrements being either also variables or constants.

The crucial point is the twofold representation of infinitesimal quantities as "zeros"("nothing else as absolute zeros") and "ciphers". Although Euler stressed the importance of this distinction to avoid "maximal confusion", people used either to ignore his warning or to overdo the importance of either the "zero" or "ciphers" and identified the meaning of "ciphers" and "zeros" replacing the former with the latter. Already in the 18th century, Euler's foundation had been declared to be the "calculus of zeros" [Lagrange, Fonctions] instead of putting also "ciphers" in their equal rights.

The analytical representation of mechanics had been developed by Lagrange [Lagrange, Mécanique] who also joined a foundartion of the calculus [Lagrange, Fonctions] and the foundation of mechanics [Lagrange, Mécanique].[69] Following Euler and Lagrange, the infinitesimal quantities had been interpreted at first mathematically and at second mechanically. In contrast to Newton's foundation who assumed the model of flowing quantities, both the interpretations are introduced independently of each other, i.e. the differentials are not interpreted in terms of fluents and fluxions, but, on the contrary, time is interpreted in terms of differentials and, consequently, mechanics turned out to be an "extension of the geometrical analysis".

Euler distinguished between finite and infinitesimal increments and, mechanically, between spatial and temporal increments whose representations by differentials are called "spatiolum" and "tempusculum", respectively (compare Chap. 4). The syntax is provided by the arithmetical rules of calculus whereas the different interpretations follow from mechanics. By the same procedure and additionally by a precise discrimination between the syntactical structures and their semantic interpretation in terms of the difference between "indeterminate" and "determinate" quantities, i.e. variables and constants, respectively [Euler E101, §§ 1–6], Euler set apart "ciphers" and "zeros" to avoid "maximal confusion" [Euler E212, § 85], [Euler 1727]. Following Euler, "ciphers" represent either function or variables

[69] "Ainsi, on peut regarder la Mécanique comme une Géométrie à quatre dimensions et l'Analyse mécanique comme une extension de l'Analyse géométrique." [Lagrange, Fonctions, Part III, § 1] Lagrange assumed as basic principles of mechanics: (i) conservation of living forces, (ii) conservation of the motion of the centre of gravity, (iii) conservation of the moments of rotation (angular momentum), (iv) principle of least action. [Lagrange, Mécanique, Seconde Partie, Dynamique, pp. 237, § 13 (p. 257)]

whereas “zero” is a “determinate” quantity since “zero” cannot be treated as a variable.

The analytical foundation of mechanics was motivated by the lack of algorithms in the geometrical representation by Newton, Hermann and other authors [Euler E015/016, Preface]. The same problem concerning the algorithms could be formulated for the calculus, but rather for the search of a reliable demonstration of the validity of the algorithms which had been invented by Leibniz and Newton in 1680 and 1687, respectively. In 1684, Leibniz published at once the whole set of algorithms which were appropriate to replace the conventional geometrical methods in mechanics by analytical representation. However, it took some decades before Euler presented the result [Euler E015/016]. Euler’s approach to mechanics was closely connected to his idea that *mechanical laws* remain to be valid if they are completely formulated in terms of relations between differentials of time and space instead of finite temporal and spatial intervals. Varignon developed such representation for uniform motion in a straight direction.

Euler claimed that there are no numbers whose multiplication results in the number equal to zero. In § 72, the increase of a quantity is discussed (augeri posse). In § 73, the series of integers 1,2,3,4,... is presented as a model of the unlimited increase of numbers which is confirmed by result of summation, in § 82, the sum $1+2+3+4+\cdots$ is investigated thereby the sign ∞ is introduced. In the following paragraphs, Euler used the signs *a, dx* and *A* for finite, infinitesimal (differentials) and infinite quantities (see below). Referring to the Leibnizian model of the transition from the ellipse to the parabola by moving one focal point to infinite (§ 82), Euler emphasized that “this theory of the infinity will be further illustrated if we discuss that which mathematicians call the infinitely small” (§ 83) All paragraphs are from [Euler E212].

Referring to Leibniz, Euler claimed that “an infinitely small quantity is nothing but a vanishing quantity, and so it is really equal to 0” or “less than any assignable quantity” [Euler E212, § 83]. However, a “vanishing” quantity or a quantity “less than any assignable one” is defined with respect to a finite quantity which had been diminished. Consequently, for the arithmetic ratio, the rules $dx = 0$ and $adx = 0$ are established, but carefully distinguished from the geometric ratio of two infinitesimal quantities *dx & dy* whose ratio is “not a ratio of equals”. On the contrary, the geometric ratio *dx/dy* can take any value and one would run into trouble and confusion if these quantities are not distinguished by the use of different signs.

Hence, the subject of the calculus is the geometric ratio of differentials *dx & dy*, i.e. *dx/dy* or *dy/dx* (compare the Preface of *Institutiones* [Euler E212, Preface] and Newton [Newton, Quadrature (Harris)][70]). Summarizing, Euler claimed that there

[70] In the contemporary interpretation, the quantities are interpreted as “differential quotients” [Klein, Elementarmathematik] either as $\mathrm{d}y/\mathrm{d}x$ or as $\mathrm{d}x/\mathrm{d}y$ if either y is a function of x, $y = y(x)$, or x is a function of y, $x = x(y)$, respectively.

are no approximations[71] at all and, ss a consequence of the introduced rules, the same rigour is attained as in the works of the Ancients.

> 87. Cum igitur infinite parvum sit revera nihil, patet quantitatem finitam neque augeri neque diminui, si ad eam infinite parvum vel addamus vel ab ea subtrahamus. (...) Hinc sequitur canon ille maxime receptus, quod *infinite parva prae finitis evanescant, atque adeo horum respectu reiici queant*. Quare illa obiectio, qua Analysis infinitorum rigorem geometricum negligere arguitur, sponte cadit, cum nil aliud reiiciatur, nisi quod revera sit nihil. Ac propterea iure affirmare licet, in hac sublimiori scientia rigorem geometricum summum, qui in Veterum libris deprehenditur, aeque diligenter observari. [Euler E212, § 87][72]

Euler's statement had been questioned by Lagrange who was not satisfied by the "calculus of zeros" [Lagrange, Fonctions] and even presented that interpretation. Euler intended to exclude in order to avoid "greatest confusion".[73] Nevertheless, as it can been shown, also Lagrange indirectly made use of Bernoulli's rule assuming that, in goal and spirit of the Ancients, the infinite represents a limit. Instead of using differentials, Lagrange performed the analytical operation with the inverse of an infinitesimal quantity.

> Because an infinite quantity cannot be increased by addition; and therefore $\infty \cdot D + C = \infty \cdot D$, and $\infty \cdot D' + C' = \infty \cdot D'$; consequently, $\frac{\infty \cdot D + C}{\infty \cdot D' + C'} = \frac{\infty \cdot D}{\infty \cdot D'} = \frac{D}{D'}$. [Euler E387, Additions by Lagrange]

Lagrange considered an infinite quantity like a limit which can be approached as close as possible, but never attained or pass through.[74] Obviously, Lagrange's rela-

[71] The interpretation in terms of approximations had been discussed by Leibniz. "(...) just I have denied of the reality of a ratio, one of whose terms is less than zero, I equally deny that there is properly speaking an infinite number, or an infinitely small number, or any infinite line or any line infinitely small. (...) The infinite, whether continuous or discrete, is not properly a unity, nor a whole, nor a quantity, and when by analogy we use it in this sense, it is a certain *facon de parler*, I should say that when a multiplicity of objects exceeds any number, we nevertheless attribute to them by analogy a number, and we call it infinite. And thus I once established that when we call an error infinitely small, we wish only to say an error less than any given, and thus nothing in reality. And when we compare an ordinary term, an infinite term, and one infinitely infinite, it is exactly as if we compare, in increasing order, the diameter of a grain of dust, the diameter of the earth, and that of the sphere of the fixed stars." [Leibniz GM V, 389]

[72] "From this we obtain the well-known rule that *the infinitely small vanishes in comparison with the finite and hence can be neglected*. For this reason the objection brought up against the analysis of the infinite, that it lacks geometric rigor, falls to the ground under its own weight, since nothing is neglected except that which is actually nothing. Hence with perfect justice we can affirm that in this sublime science we keep the same perfect geometric rigor that is found in the books of the ancients." [Euler E212, § 87]

[73] "Maclaurin et d'Alembert emploient la considération des limites et regardent le rapport des différentielles comme le limite du rapport des différences finies, lorsque ces différences deviennent nulles. Cette manière de représenter les quantités différentielles ne fait que reculer la difficulté; car, en dernière analyse, le rapport des différences évanouissantes se réduit encore à celui de zéro à zéro. (...) Les véritables limites, suivant les notions des anciens, son des quantités qu'on ne peut passe, quoiqu'on puisse s'en approcher aussi près que l'on veut; telle est, par exemple, le circonférence du cercle à l'égard des polygones inscrit e circonscrit, parce, que, quelque grand que devienne le nombre des côtés, jamais le polygone intérieur ne sortira du cercle, ne l'extérieur n'y entrera." [Lagrange, Fonctions]

[74] Geometrically, the circle is of finite extension. Hence, the infinite component can be only introduced *arithmetically* by the increase of the number of sides of polygons and, simultaneously, the decrease of the lengths of the sides of the inscribed and circumscribed polygons.

tion $\infty \cdot D + C = \infty \cdot D$ is the complement to Bernoulli's rule $a \pm dx = a$ [Bernoulli 1691–1692], [Euler 1727]. Bernoulli's rule is easily obtained by dividing the relation by the infinite number labelled by ∞ if $C/\infty = dx$ is assumed for the inverse operation.

In generalized version, Bernoulli's rule[75] $a \pm o - a = 0$ had been given by Euler

$$a \pm ndx - a = 0 \tag{5.24}$$

which gives straightforwardly

$$\frac{a \pm ndx}{a} = 1 \quad \text{and} \quad \frac{A \pm n \cdot a}{A} = 1 \tag{5.25}$$

in Euler's and Lagrange's representation, respectively. Nevertheless, the argumentation is different. Euler claimed that a finite quantity cannot increased by addition of an infinitesimal quantity[76] whereas Lagrange claimed that the infinite cannot be increased at all.

Then in the following paragraphs, Euler completed the set of algorithms. The relations between infinitesimal quantities of different orders of magnitude are defined by the same relations as "evanescent quantities" (in Euler's interpretation) which had been introduced before, i.e. "dx^2 prae ipsa dx evanescit" [Euler E212, § 88]. Consequently, it holds $dx \pm dx^2 = dx$ and, therefore, it follows

$$dx \pm dx^2 : dx = \frac{dx \pm dx^2}{dx} = 1 \pm dx = 1 \tag{5.26}$$

$$dx \pm dx^{n+1} : dx = 1 + dx^n \tag{5.27}$$

in case of exponents being integers. For integers of different magnitude and for exponents being rational number, the relations

$$adx^m + bdx^n = adx^m \text{ for } m < n \tag{5.28}$$

and

$$\sqrt{dx} = dx^{\frac{1}{2}} \quad \text{and} \quad a\sqrt{dx} + bdx = a\sqrt{dx} \tag{5.29}$$

are derived, respectively. [Euler E212, § 89]

In the following, Euler discussed analytical expressions composed of finite, infinitesimal and infinite quantities [Euler E212, §§ 90–97].

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75 "(...) alia vero insuper adjicienda esse, que ex conditione incrementum infinite parvorum fluant (...). Eorum autem, quae ex incrementorum infinitorum parvitate oriuntur, commune principium hoc est. Quantitas quaecunque addita vel subtractione aliorum quantitatum quae respectu ipsius sunt infinite parva, neque augetur, neque minuitur: Nam si augeretur vel minueretur, eae quantitate quae additae vel ablatae sunt, ad eam rationem assignabilem haberent, ad infinitam quod esse contra hypothesin. Unde sequitur, quantitates infinite parvae respectu finitarum rejici passe. Erit ergo $x \pm o = x$ si o fuerit infinito parvo ratione ipsius x." [Euler 1727]

76 "(...) quod infinite parva prae finitis evanescat, atque adeo horum respecta reiici queant." [Euler E212]

> Namque quo maior statuitur fractionis $\frac{a}{z}$ denominator z, eo minor sit fractionis valor, atque si z fiat quantitas infinite magna seu $z = \infty$, necesse est, ut fractionis valor $\frac{a}{\infty}$ fiat infinite parvus. [Euler E212, § 90]

Obviously, the relations

$$\frac{a}{dx} = \infty \quad \text{and} \quad \frac{a}{\infty} = dx \tag{5.30}$$

are obtained by the application of the arithmetical operations. Euler's comment, however, is misleading.[77] This misleading elaboration had been corrected in § 95. The following relations are presented [Euler E212, § 95] where A denotes an infinite quantity.

$$\frac{a}{dx} = A, \frac{a}{dx} : \frac{A}{dx} = a : A \text{ and } \frac{a}{dx^2} = \frac{A}{dx} \tag{5.31}$$

In § 93 Euler discussed infinite quantities of different magnitude and the difference between two infinite quantities of different magnitude.

> 93. (...) sequitur inter quantitates infinite magnas rationum quamcunque locum habere posse. Hincque, si quantitas infinita per numerum finitum sive multiplicetur, sive dividatur, prodibit quantitas infinita.[78] Neque ergo de quantitatibus infinitis negari potest, eas ulterius augeri posse. Facile autem perspicitur, si ratio geometrica, quam duae quantitates infinitae inter se tenent, non fuerit aequalitatis, multo minus earum rationem arithmeticam aequalitatis esse posse, cum potius earum differentia semper sit infinite magna. [Euler E212, § 93]

Obviously, this theorem is correlated to the corresponding geometric ratio of two infinitesimal quantities [Euler E212, § 85]. For two infinitely large quantities, it follows

> Ponatur quantitas illa infinita, quae ex divisione quantitatis finitae a per infinitae parvam dx oritur, $= A$, ita ut sit $\frac{a}{dx} = A$: erit utique $\frac{2\cdot a}{dx} = 2 \cdot A$ & $\frac{n\cdot a}{dx} = n \cdot A$; cum igitur & $n \cdot A$ sit quantitas infinita, sequitur inter quantitates infinite magnas rationem quamcunque (...). [Euler E212, § 93]

Then, it follows

> Simili modo erit a $\frac{a}{dx^3}$ quatitas infinita infinities maior quam $\frac{a}{dx^2}$, ideoque infinita infinites maior quam $\frac{a}{dx}$. [Euler E212, § 95]

Consequently, the different orders of infinity had to be distinguished from each other and demonstrated by the expressions $\frac{a}{dx}dx = a, \frac{a}{dx}dx^2 = adx, a, adx, adx^2, adx^3,$ & c.

> § 97. His de gradibus infinitorum praemonitis, mox apparebit fieri posse, ut productum ex quantitate infinite magna in infinite parvam non solum quantitatem finitam producat, quod supra evenisse vidimus; sed etiam huiusmodi productum esse poterit sive infinite magnum sive infinite parvum. [Euler E212, § 97]

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[77] In the second relation and in the following §§ 91 and 92, Euler added $dx = 0$ without having a finite quantity for comparison. Such parts of the text may have caused misinterpretations of Euler's approach since Euler made use of the sign "0" instead of the signs for infinitesimal quantities introduced before.

[78] This had been also assumed by Lagrange, but, in contrast to Euler, the infinite is treated the as an invariant limit, i.e. as a quantity which cannot be modified, i.e. neither increased nor decreased.

Finally, the results are summarized in the general formula [Euler E212, § 98].

$$(a/dx^n)bdx^m = a \cdot b \cdot dx^{m-n} \tag{5.32}$$

The outcome depends on the magnitude and te sign of the difference $m-n$: (i) infinitely small for $m>n$, (ii) finite for $m=n$ and (iii) infinitely large for $m<n$. Here, the full power of Euler's analytical approach is demonstrated since there are neither remaining indeterminate nor hidden elements in the foundation of the algorithms. Furthermore, in Eq. (5.32) the differentials dx^m and dx^n are introduced on an equal footing with the finite quantities a and b as quantities of different type.

The only remaining point which had been left out by Euler is the interpretation of Bernoulli's rule $a \pm ndx - a = 0$ for the case $a=0$. Euler argued that the result $0 \pm ndx - 0 = \pm n \cdot dx = 0$ is correct since $dx = 0$. However, this conclusion is only correct if the *infinitesimal* quantity is in an *arithmetic ratio* to any *finite* quantity [Euler E212, §§ 84–86] or $x = const$, but fails if the infinitesimal quantity dx is in an arithmetic ratio to the real number zero 0 being the "only infinitesimal real number". Then, the relation $0 \pm ndx - 0 = \pm n \cdot dx = 0$ cannot be valid since $+n \cdot dx \neq -n \cdot dx$ if the sign are interpreted by the same rules being valid for finite quantities, i.e. $+n \neq -n$. Consequently, for $x \neq const$ (i) $+n \cdot dx \neq 0$ and $-n \cdot dx \neq 0$ are different from zero, but preserve their status (ii) to be "less than any assignable quantity" or, less than any finite quantity. Relying upon and keeping in mind Euler's warning

> 85. (...) In the calculus of the infinitely small, we deal precisely with geometric ratios of infinitely small quantities. For this reason, in the calculations, unless we use different symbols to represent these quantities, we will fall into the greatest confusion with no way to extricate ourselves. [Euler E212, § 85]

we have to distinguish the "real zero" and the "differentials" or "infinitely small quantities" not only by different signs, but also by their different status. Like the imaginary numbers which are to be a special kind of numbers, the differentials are also a special kind of numbers since all arithmetical operation defined for integers and real number remain to be valid for differentials. As numbers, the differentials exhibit the same strange behavior as other kind of numbers being different from integers and rational numbers. Euler commented on imaginary numbers, they "must belong to an entirely distinct species of numbers; since they cannot be ranked either among positive or among negative numbers" [Euler E387, § 141] For the differentials having different signs, following Leibniz [Leibniz, Nova methodus], the relations (i) $d\bar{x} = d(x) = dx$ and (ii) $d\overline{-x} = d(-x) = -dx$ are implicitly assumed to be valid. Then, the inequalities

$$-a < dx < +a \quad \text{and} \quad -a < -dx < +a \tag{5.33}$$

are simultaneously to be fulfilled for any finite number a. Following Euler and replacing the differential dx by the number zero,[79] these relation degenerate since the distinction between *positive* and *negative* numbers is expunged

[79] "83. (...) Likewise there is a definition of the infinitely small in the same mind (as the definition of the infinite) where it will be said that it is less than any assignable quantity: If a quantity is so small that it is less than any assignable quantity, than it cannot not be 0 (nulla, nothing), since unless it is equal to 0 an equal quantity might be able to be assigned to himself, and this contradicts

$$-a < 0 < +a \quad \text{and} \quad -a < 0 < +a \tag{5.34}$$

and only *one* relation instead of two inequalities is obtained. Nevertheless, this relation is valid for all numbers of *finite* magnitude (being chosen as small as desired). With respect to the number zero, the inequality is symmetric whereas both the inequalities for differentials are asymmetric in all cases where dx had to be treated as a "cipher" [Euler E212, § 84]. Denoting a infinitesimal quantity by $\varepsilon \approx 0$ [Keisler] the relations

$$-a < \varepsilon < +a \quad \text{and} \quad -a < -\varepsilon < +a \tag{5.35}$$

can be introduced and the infinitesimal quantity is defined as a "quantity being different from zero, but less than any finite real number" [Keisler]. This property had been implicitly introduced by Leibniz who assumed that there are differentials of positive as well as of negative signs [Leibniz, Elementa], [Leibniz, Nova methodus].

Obviously, Newton would run into trouble assuming that the "only infinitely little quantity *o*" is capable to have different sign, i.e. the continuous flow can change its direction. Therefore, Newton introduced moments made up of the product of velocities and the "infinitely little quantity *o*".

5.4 Reconsideration of the Calculus: Robinson

Euler generalized the Leibnizian principle of continuity and that it is also valid for series of discrete quantities like the natural numbers or the natural numbers including negative numbers and, furthermore, rational numbers [Euler E212, § 85]. In the 20th century, also referring to Leibniz's *principle of continuity*, Robinson claimed that "Leibniz was on the right track" since the principle of continuity is a forerunner of the *transfer principle*. Leibniz stated:

> For I have, beside the mathematical infinitesimal calculus, a method also for use in Physics [Specimen dynamicum, 1695], (...) and both of these I include under the Law of Continuity; and adhering to this, I have shown that the rules of the renowned philosophers Descartes and Malebranche were sufficient in themselves to attack all problems of motion. I take for granted the following postulate:
>
> *In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.* [Leibniz, Specimen, II (4) (Keisler)]

Keisler commented:

> Leibniz rather consistently favoured the infinitesimal method but believed (correctly) that the same results could be obtained using only real numbers. He regarded the infinitesimals as 'ideal' numbers like the imaginary numbers. To justify them he proposed his law of

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the hypothesis. (...). To anyone who asks what an infinitely small quantity in mathematics is, we can respond that it is really equal to zero. Nor therefore in this idea such a great mystery is hidden (...). There is really not such a great mystery lurking in this idea as some commonly think and thus have rendered the calculus of the infinitely small suspect to so many." [Euler E212, § 83]

> continuity (...). This 'law' is far too imprecise by present standards. But he was a remarkable forerunner of the Transfer Principle on which modern infinitesimal calculus is based. Leibniz was on the right track, but 300 years too soon! [Keisler, Elementary calculus, 2000]

The *transfer* principle is completed by the *extension* principle and the *standard part* principle [Keisler].[80] Non-standard analysis is based on these principles. The rules of the calculus are derived by the application of analytical and arithmetical demonstrations without direct reference to geometry.

In goal and spirit, Euler's program is similar to Leibniz's idea to develop a method called *Characteristica universalis* [Leibniz, Fragmente] which is based on joining formal and algorithmic approaches to make the process of thinking and demonstration reliable and feasible. Leibniz only partially succeeded to put the program into practice, but his main discoveries, the *analytical* representation of the *calculus* and the system of *dual numbers* have influenced the scientific development over the next centuries. The representation of number (integers) by *different systems of signs* (symbols or formalized systems of symbols) preserving the arithmetical operations is also of great methodological importance for mechanics (compare Chaps. 7 and 8). Euler made use of different representations of the calculus in terms of (i) ciphers and (ii) numbers [Euler E212] being appropriated for formal operation based on algorithms and the numerical representation of the results, respectively. Distinguishing between these types of quantities, Euler removed the inconsistency mentioned by Nieuwentijt [Nieuwentijt, Analysis] and later by Berkeley [Berkeley, Analyst] that the differentials play a double role being (a) either *different from zero* and equal or (b) *equivalent or equal to zero*. The property (a) is demonstrated by the internal relations between the powers of differentials represented by the relation $dx > dxdx > dxdxdx > \dots$ and the powers of zero represented by the relation $0 = 0 \cdot 0 = 0 \cdot 0 \cdot 0 = \dots$ which result from the *comparison of infinitesimal* quantities based on arithmetics. This relation is an *internal* relation defined for infinitesimal quantities whereas the relation between a finite quantity and an infinitesimal quantity is an *external* relation between quantities of different kind. Nowadays, this relation had been analyzed in terms of Leibniz's theorem on the *syncategorematic* nature of differentials (compare Arthur [Arthur, Synacategorematic]). Syncategorematic does not only mean that the differentials not only have to be defined by their *internal* relations as quantities of different magnitude, but also means that there is an *external* relation between finite and infinitesimal quantities.

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[80] "*Extension principle*. (a) The real numbers form a subset of the hyperreal numbers, and the order relation $x > y$ for real numbers is a subset of the order relation for hyperreal numbers. (b) There is a hyperreal number that is greater than zero but less then every positive real number.(c) For every real function f of one ore more variables we are given a corresponding hyperreal function f^* of the same number of variables. f^* is called the natural extension of f.

Transfer principle. Every real statement that holds for one or more particular real functions holds for the hyperreal natural extensions of these functions.

Standard part principle. Every finite hyperreal number is infinitely close to exactly one real number.

Let b be a finite hyperreal number. The standard part of *b*, denoted by *st(b)*, is the real number which is infinitely close to *b*. Infinite hyperreal numbers do not have standard parts." [Keisler]

The *only real infinitesimal quantity* zero is not necessarily a term in a series of *rational numbers*, but only necessarily a term in the series of *positive and negative integers* (compare Chap. 8). Hence, the number zero is *external* with respect to a series $\cdots - 3/2, -1/2, +1/2, +3/2, \cdots$, i.e. it has to be additionally defined or is only obtained for the 2nd difference of the terms of the series. Although it is not introduced as a term of the series, but definitely *excluded*, it represents an invariant relation between the terms of the series. This relation is independent of the magnitude of the terms. Each term of the series can be increased or decreased by the multiplication with a *finite scaling factor a*. The result is

$$\cdots - 3a/2, -1a/2, +1a/2, +3a/2, \cdots \tag{5.36}$$

but the invariant property is preserved for different interpretations of the quantity a. Hence, it follows $-a/2 < 0 < +a/2$. In case of differentials, we obtain

$$\cdots - 3da/2, -1da/2, +1da/2, +3da/2, \cdots \tag{5.37}$$

and, as before in case of finite a, it follows $. - da/2 < 0 < +da/2$. Consequently, each of the terms in (5.36) and (5.37) has to be different from zero since only the 2nd differences are identically zero.

Hence, the series can be only analytically interpreted in terms of increments if the increment is different from zero. In Keisler's notation for finite, infinitesimal and infinite numbers, it follows

$$\frac{a}{H_1} = \varepsilon_1 \qquad \text{and} \qquad \frac{a}{H_2} = \varepsilon_2, \tag{5.38}$$

where the infinitesimal numbers ε_1 and ε_2 are of different magnitude, i.e. either $\varepsilon_1 > \varepsilon_2$ or $\varepsilon_1 < \varepsilon_2$, being related to infinite numbers H_1 and H_2 of different magnitude. The same result is obtained for different infinite number being derived from different infinitesimal numbers by the inverse relations

$$\frac{a}{\varepsilon_1} = H_1 \qquad \text{and} \qquad \frac{a}{\varepsilon_2} = H_2. \tag{5.39}$$

Alternatively, the relations can be written as follows

$$a = \varepsilon \cdot H = \varepsilon \frac{a}{\varepsilon} = \varepsilon \cdot \frac{1}{\varepsilon} \cdot a = a. \tag{5.40}$$

The statement of Euler "Indeed, the larger the denominator z of the fraction a/z becomes, the smaller the value of the fraction becomes, and if z becomes an infinitely large quantity, that is $z = \infty$, then necessarily the value of the fraction a/∞ becomes infinitely small." [Euler E212, § 90] can be expressed in terms of hyperreal numbers by the relation $\varepsilon = a/\mathrm{H}$ [Keisler].

Almost not surprisingly, Euler also made use of a generalized version of the Leibnizian principle of continuity and assumed it to be valid also for series of discrete quantities like the natural numbers or the natural numbers including negative numbers and, furthermore, rational numbers [Euler E212, § 85]. Furthermore, generalizing Leibniz's approach to the calculus of differences and sums, Euler transferred the validity of all arithmetical operations to numbers being different from finite numbers. Hence, also Euler "was on the right track" and created a reliable basis for the further development of the calculus and its application to mechanics.

Chapter 6
Euler's Early Relativistic Theory

77. Die scheinbare Bewegung bezieht sich auf einen Zuschauer (...)
Von dieser (der scheinbaren) Bewegung ist um so viel nöthiger hier zu handeln, da wir uns in der Welt keinen anderen Begriff als von der scheinbaren Bewegung machen können; denn wir können die Oerter der Körper nicht anders als nach dem Orte unseres Aufenthaltes schätzen.

[Euler E842, § 77][1]

In this Chapter we will analyze Euler's contributions to mechanics which may be considered as a continuation and specification of Leibniz's criticism of Newton's theory of absolute time, space and motion. However, in contrast to Leibniz, Euler maintained Newton's concept of absolute time and space, but rejected the idea of absolute motion [Euler E015/016, §§ 7 and 97]. Nevertheless, related to the later 19th century criticism of Newton's theory by Mach and to the new ideas in the 20th century by Einstein, Euler anticipated essential features of the advanced theory of relativity, presently known from the textbooks. Although Euler's comprehensive treatise entitled *Anleitung zur Naturlehre* had been lastly published with a delay of more than hundred years in 1862 [Euler E842],[2] its impact on the rise of relativism in the 19th century was negligible.[3] The overwhelming success of his mathematical writings might have obscured his fundamental contributions to physics, but as another reason it is also likely that Euler's foundation of mechanics was not fully understood by his contemporaries nor by the followers in the 19th century. Mach published his fundamental analysis not before the beginning of the 80th and mentioned the unpleasant response of the physics community on earlier attempts

[1] "77. *The apparent motion is related to an observer* (...). In any case and for principal reasons, we are in great need to consider apparent motion since we cannot get another idea as of apparent motion if we are in the world, since we are not able to estimate the positions of the body in another way except according to the place of our stay (...)." [Euler E842, § 77]

[2] Originally published in *Opera Postuma* 2, 1862, pp. 449–560, *Opera Omnia*: Series 3, Volume 1, pp. 16–180.

[3] Mach published his criticism of absolute time and space in the book entitled *Mechanik in ihrer Entwicklung* twenty years later in 1883 [Mach, Mechanik]. Nowadays, Euler's theory had been analyzed by Maltese [Maltese].

D. Suisky, *Euler as Physicist*, DOI 10.1007/978-3-540-74865-6_6,

of the "relativists" [Mach, Mechanik]. Hence, even though Euler analyzed mechanics with principles which were maintained by Mach and Einstein, neither Mach [Mach, Mechanik] nor Helmholtz nor Einstein referred explicitly to Euler. Nevertheless, Euler demonstrated all advantages of the analytical approach not only in solving traditional problems, but also in posting completely new questions which could not even be formulated without such a kind of theory of motion. This will be demonstrated for Euler's analysis of the invariance of the equation of motion given in the treatise *Anleitung zur Naturlehre* [Euler E842].

A treatise commensurable in goal and spirit with Euler's *Mechanica* (1736) and *Theoria* (1765) and d'Alembert's *Traité* (1743) was only published by Lagrange entitled *Mécanique analytique* in 1788.[4] However, the comprehensive summary in the *Anleitung zur Naturlehre* was only published posthumously in 1862. Although it was written in German and Euler already included essential parts in *Mechanica* and *Theoria* which had been translated into German by Wolfers[5] [Euler E015/015 (Wolfers)], [Euler E289 (Wolfers)] in 1848 and 1853 the impact on German writing scholars like Mach was negligible (compare [Mach, Mechanik]). Most likely, Mach underestimated the power and the prospects of the analytical form.[6]

In the second half of 19th century [Helmholtz, Vorlesungen] and in the end of the 20th century,[7] a revival of the 18th century debate on the foundation of mechanics can be noticed which is centred upon the discussion of the methodological and logical interconnection of mechanics with the other parts of the legacy of Newton and Leibniz (to describe this renaissance of the 17th and 18th century discussion on the basic concepts of time, space and motion an interesting distinction between *mathematical* and *theoretical* physics had been made).[8] Helmholtz referred explicitly to

[4] "One will not find figures in this work. The methods that I expound require neither constructions, nor geometrical or mechanical arguments, but only algebraic operations, subject to a regular and uniform course." [Lagrange, Mécanique]

[5] Wolfers translated also Newton's *Principia* into German (1872).

[6] "Nach ihm [Newton] ist ein wesentlich neues Prinzip nicht mehr ausgesprochen worden. Was nach ihm in der Mechanik geleistet worden ist, bezog sich durchaus auf die deduktive, formelle und mathematische Entwicklung der Mechanik auf Grund der Newtonschen Prinzipien." [Mach, Mechanik, p. 179]

[7] Chandrasekhar [Chandrasekhar], Wilczek [Wilczek 2004a], Smolin [Smolin].

[8] "A key point of the paper is the difference in approach to physical problems taken by mathematical physicists as opposed to theoretical physicists. In a paper published in 1908 Minkowski reformulated *Einstein*'s 1905 paper by introducing the four-dimensional (space-time) non-Euclidean geometry, a step which *Einstein* did not think much of at the time. But more important is the attitude or philosophy that Minkowski, *Hilbert* – with whom Minkowski worked for a few years – *Felix Klein* and *Hermann Weyl* pursued, namely, that purely mathematical considerations, including harmony and elegance of ideas, should dominate in embracing new physical facts. Mathematics so to speak was to be master and physical theory could be made to bow to the master. Put otherwise, theoretical physics was a subdomain of mathematical physics, which in turn was a subdiscipline of pure mathematics. In this view Minkowski followed *Poincaré* whose philosophy was that mathematical physics, as opposed to theoretical physics, can furnish new physical principles. This philosophy would seem to be a carry-over (modified of course) from the Eighteenth Century view that the world is designed mathematically and hence that the world must obey principles and laws which mathematicians uncover, such as the principle of least action of *Maupertuis, Lagrange* and

Leibniz for the conservation of energy (Leibniz's "living forces" or Helmholtz's "Die Erhaltung der Kraft" [Helmholtz, Kraft]),[9] and later, in 1920, Reichenbach acknowledged Leibniz's theory, but neither credited Euler. Essential elements of the relativistic approach had already been published in the *Mechanica* (1736) and the *Theoria* (1765) (compare Chap. 4) becoming standard textbooks in the second half of the 18th century [Kästner, Anfangsgründe],[10] [Oberwolfach].[11] In the end of the 19th century, the development of relativism was represented by different scholars being disciples of Mach. However, there was no unified approach since Mach preferentially discussed the experimental basis of relativism whereas Helmholtz developed in first respect the theoretical background of Newton's mechanics without reference to Mach's relativistic approach.[12] This approach had been completed by

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Hamilton. Einstein was a theoretical physicist and for him mathematics must be suited to the physics." [L. Pyenson, Hermann Minkowski and Einstein's Special Theory of Relativity: With an appendix of Minkowski's 'Funktiontheorie' manuscript, Arch. History Exact Sci. **17** (1) (1977), 71–95]. In view of this analysis and classification, Euler had to be considered as a scholar who was unifying the abilities and methods of a *mathematical* and *theoretical* physicist and, moreover, being a mathematician.

[9] The rediscovery of Leibniz's contributions to mathematics and logic by Gerhardt [Gerhardt, Leibniz], [Gerhardt, Historia] Couturat [Couturat, Opuscules] and Russell [Russell, Western] had been already commented in Chaps. 2 and 3. The rediscovery of Leibniz's ideas had been continued also in the 20th century [Reichenbach], and is still progressing [Leibniz Edition, BBAW]. Nowaday, Leibniz's contributions to mathematics are highly acknowledged by mathematicians [Mandelbrot], [Robinson], physicists [Smolin], philosophers and historians of science (Arthur), [Meli], [Knobloch, Rigorous], [Knobloch, Parmentier]. [http://prof.mt.tama.hosei.ac.jp/∼hhirano/academia/leibniz.htm]

[10] Kästner pointed out that he followed in his treatise the procedure Euler had given in the *Mechanica*. "Da ich aus Hrn. Eulers Buche, als es noch ganz neu war, diesen Theil der höheren Mechanik durch eigenen Fleiß erlernt (...)." [Kästner, Anfangsgründe, Vorwort]

[11] Recently, the broad spectrum of the reaction on Euler's legacy had been informative reviewed: *The reception of the work of Leonhard Euler* (1707–1783), Workshop held in Oberwolfach, August 12–18, 2007, organized by Ivor Grattan-Guinness and Helmut Pulte. Algebra and analysis (Thiele), Series (in the early 19th century) (Jahnke), Reception of treatises on Calculus (Domingues), Calculus of variations (Nakane), Space and time (Pulte), Celestial mechanics (Wilson), Elasticity, letters from Legendre to Sophie Germain (Langton), Hydrodynamics (Mikhailov), Engineering (Sandifer), Approximation Theory (Steffens), Wave Theory of Light (Home), Electricity and Magnetism (Radelet-de Grave), Algebra and number theory (Neumann), Lettres à une Princesse d'Allemagne (Breidert), Reception by the French (Grattan-Guinness).

[12] The absence of any remark on Mach in Helmholtz's *Vorlesungen* demonstrates the novelty of the consequent relativistic approach based on the criticism of Newton's assumptions on space and time. Obviously, the relativistic approach is automatically excluded if mechanics is based on the theorems valid for a mass point. "Erster Theil. Kinematik eines materiellen Punktes, §§ 1–7. Zweiter Theil. Dynamik eines materiellen Punktes. §§ 8–38. Dritter Theil. Dynamik eines Massensystems. Erster Abschnitt. Das Reactionsprincip. §§ 39–51. Zweiter Abschnitt. Das Energieprincip. §§ 48–52 Dritter Abschnitt. Anwendung. Die Bewegung der Himmelskörper. §§ 53–57." [Helmholtz, Vorlesungen] The "reaction principle" cannot be properly defined for a single mass point (a single body). "39. Newtons drittes Axiom. In den vorangehenden Betrachtungen beschäftigte uns die theoretische Ableitung des Bewegungsgesetzes eines einzelnen Massenpunktes, auf welchen gewisse von außen gegebene Kräfte einwirken. Wir gehen nun dazu über, gleichzeitig mehrere Massenpunkte zu betrachten." [Helmholtz, Vorlesungen, § 39] Euler entitled the first Chapter of the *Mechanica* "De motu in genere" and discussed the frame of reference

Leibniz's conservation law for "living forces" (see Chap. 7), but without reference to the theoretical confirmation of the invariance of the equation of motion in inertial systems. This law was implicitly assumed by the Galilean discovery that only the change in motion is needed to be explained, but not the conservation of uniform motion, i.e. for the *absence* of forces.[13] Adding the requirement of invariance for the 2nd Law, i.e. for the *presence* of forces, Euler completed the Galilean law (Newton's 1st axiom) by the discovery that only in non-accelerated frames of reference the acting forces are invariant.[14]

This part of Euler's mechanics which may be called an early relativistic theory results directly from his program for mechanics from 1736 [Euler E016/016] (compare Chap. 4) which is closely connected to the foundation of the calculus [Euler, E212] (compare Chap. 5). The invariance of the equation of motion for Galileo transformation is probed for a representation in terms of differentials (see Sect. 6.3.4). Euler demonstrated the power of analytical approach by unifying the apparently conflicting approaches of his predecessors Newton and Leibniz assuming that (i) time and space play the role *absolute* quantities [Euler E149] whereas motion is preferentially defined as a *relational* notion [Euler E842].[15] Thirty years later between 1760 and 1765,[16] the theory of the motion of rigid bodies [Euler E289] is also presented as a special kind of relative motion, i.e. as the motion of an extended body relatively to a fixed point or a fixed axis [Euler E289]. Hence, Euler's theory of relative motion had been scattered over different treatises published between 1736

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for the investigated body formed by other bodies [Euler E015/016, §§ 6 and 13], i.e. introduced the *relative motion* as fundamental concepts of mechanics which is treated in parallel to Newton's concept of absolute motion in the following paragraphs. In the translation of Euler's *Theoria*, Wolfers entited the introductory part as being related to "points", i.e. "Einleitung enthaltend nothwendige Erläuterungen und Zugaben zur Bewegung von Punkten" [Euler E289 (Wolfers)], which had been later reduced to the "motion of *one* mass point".

[13] "Nun war es der große Fortschritt, welchen *Galilei* gegenüber allen früheren Naturforschern seit Aristoteles machte, dass er als die eine directe Ursache fordernde Bewegungserscheinung die Beschleunigung erkannte." [Helmholtz, Vorlesungen, § 8]

[14] Helmholtz did not discuss rest and motion as "relative rest and relative motion", but rather as "absolute rest and absolute motion", i.e. rest and motion are related to *one* mass. "Daß aber eine Masse, welche eine Zeit lang in Ruhe existirt hat und dann plötzlich anfängt, sich zu bewegen, durch irgendeine vorher fehlende Ursache dazu angetrieben werden muß, das liegt schon in der Allgemeingültigkeit des Causalitätsgesetzes, welches wir als Anschauungsform mitbringen." [Helmholtz, Vorlesungen, § 8] Here, Helmholtz replaced the Leibnizian "principle of sufficient reason" by its modified Kantian version of the "Causalitätsgesetz als Anschauungsform" whose original Leibnizian version had been used for the same purpose by Euler: "Wo eine Veränderung vorgeht, da muss auch eine Ursache sein, welche dieselbe hervorbringt, weil gewiss ist, dass nichts ohne einen zureichenden Grund geschehen kann." [Euler E842, § 1]. Currently, after a successful development of the theory of relativity, a renaissance of the Leibnizian version of the principle of sufficient reason and other Leibnizian principles can be observed [Smolin].

[15] Hence, Euler merged Newton's theory of *absolute* time and space and Descartes' and Leibniz's theory of *relative* motion. Euler formulated the Galilean principle of uniform motion *analytically* and obtained a relation being nowadays known as Galileo transformation which had been only replaced with the Lorentz transformation by Einstein in 1905 [Einstein, Bewegte].

[16] A preliminary version of the theory of rigid body motion had been published in the paper entitled *Découverte d'un nouveau principe de mécanique* presented 1750, published 1752 [Euler E177].

and 1862. Not surprisingly, the contemporaries and followers did not catch the full content and even in the 20th century some authors did not follow Euler's distinction between (i) absolute time and space and (ii) absolute motion and thought that Euler is an adherent Newton's theory of absolute motion [Friedman].

Euler continued the development which had been invented by Descartes in the 17th century. Descartes made use of the concept of the plenum which is related to the definition of the body as *res extensa sive corpus* [Descartes, Principles]. Descartes' construction had been replaced by Newton who cleared up the world from bodies preserving the extension. The procedure ended up in the concept of absolute space which is nothing else than Descartes' plenum interpreted in terms of a new type of extension.[17] The novelty Newton introduced is the *absolute time*. In the 20th century, based on the relativity of motion [Mach], [Einstein, Bewegte], both constructions are rejected by Einstein in 1905. Einstein based his analysis on the notion of simultaneity of events which is observed experimentally and defined theoretically using the light velocity as (i) an upper limit for the motions of bodies and (ii) the only velocity the light can attain traversing space, i.e. spatial intervals. The unification of space and time by the introduction of an internal measure result in the *absolute world*.[18] Here, "absolute" is defined as the absence of and dismiss from an *external* measure. The light velocity plays the role of an *internal* measure causing an order between the events. This order automatically disappears if the light velocity is removed and Minkowski's continuum of event split into the spatial and the temporal continuum. Space and time become independent of each other. Interpreted in term of continua being free of an *internal* measure, Newton's absolute space and absolute time are obtained, whereas, assuming the interpretation in terms of different orders, the Leibnizian construction of the successions and coexistences [Leibniz, Initia], [Leibniz Clarke] are generated. As far as the internal measure of order is missing, Leibniz introduced the monads [Leibniz, Monadology, § 8] instead

[17] Newton defined the absolute space as a mathematical notion. However, considering the relation to Descartes' plenum, Newton's absolute space is a development of Descartes' idea of the space filled with matter, bodies, being different from mere extension.

[18] After the raise of relativisms inaugurated by Mach in the 19th century resulting in discredit of Newton's absolute space and time, Newton's construction had been renewed by Minkowski who declared the relativistic interpreted space-time continuum of events to be the absolute world [Minkowski, Space and Time]. Descartes based the theory on the plenum or the continuum of bodies, Newton added the continuum of time and Minkowski assumed the continuum of events. Obviously, Minkowski do not need really bodies to make an event to be a part of the continuum. "The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. (...) Since the [relativity] postulate comes to mean that only the four-dimensional world in space and time is given by phenomena, but that the projection in space and in time may still be undertaken with a certain degree of freedom, I prefer to call it the *postulate of the absolute world*." [Minkowski, Space Time, p. 83] The meaning of Minkowski's definition is that the world is characterized by its internal measure. In classical mechanics, the internal measure is represented by the uniform motion [Euler E149, §§ 19–21]. Furthermore, following in goal an spirit Lagrange, Minkowski developed mechanics as a "geometry of four dimensions". "Ainsi, on peut regarder la Mécanique comme une Géométrie à quatre dimensions et l'Analyse mécanique comme une extension de l'Analyse géométrique." [Lagrange, Fonctions, Part III, § 1]

of an order represented quantitatively by numbers [Leibniz Clarke] (compare Sect. 2.2.2).

Euler introduced the following essential topics which make his mechanics different from the theories of his predecessors, (i) the rigorous statement on the priority of relative motion, combined with the introduction of an *observer* [Euler E015/016, §§ 7, 80 and 97] in the preceding paragraph of his program for mechanics [Euler E015/016, § 98], comprehensively elaborated in the *Anleitung* where the observer is called *Zuschauer* [Euler E842, §§ 77–83] and maintained in the *Theoria* [Euler E289, §§ 1–11], (ii) the introduction of more than one observer[19] who are comparing the results of their observations, which results in the confirmation of (iii) the invariance of the equation of motion,[20] (iv) the explanation of the origin of forces[21] (compare Chap. 4) and (v) the harmony between mathematics and mechanics resulting from Euler's procedure to coordinate the principles of mathematics with theorems of physics.

Leibniz's relational definition of time and space [Leibniz, Initia], [Leibniz Carke] is known to be essentially different and in strong contrast to Newton's theory of absolute time and absolute space [Newton, Principia]. In 19th century, Mach's well-known criticism ends up in the complete rejection of these basic notions of Newton's theory. However, Mach did not mention the contribution of Leibniz as a predecessor in the criticism of that part of Newton's theory. He also did not discuss the relation between absolute and relative times and spaces which have been introduced in the *Principia*. After the advent of Einstein's theory of relativity it was extremely difficult to reconstruct the sophisticated details of the Newton – Leibniz controversy since people declared Leibniz to be the winner. In 1924 Reichenbach stated that the success of Newton's *Principia* has hampered the development of mechanics for 200 years [Reichenbach].

In this Chapter we will demonstrate that Euler discussed both the parts of Newton's theory, on the one hand, the absolute time and the absolute space and, on the other hand, the relative time and the relative space. Despite the final rejection of absolute motion, only the consideration for both parts of Newton's theory allows for a complete formulation of the space-time problem. The reason is that any physical theory can be considered as composed of *absolute* and *relative* quantities (for Newton's theory compare Chap. 2). However, the kind of such quantities has been considerably changed. Nowadays, instead of time or space the *velocity of light* is considered as independent of the state of motion of bodies playing the role of an absolute quantity. Another prototype of such fundamental constants is *Planck's action parameter* as far as quantum theory is concerned. Newton mentioned that the relative time is "a measure of duration by means of motion". Therefore, the whole

[19] Usually, the introduction of observers is not mentioned in literature. It is impossible to assume absolute motion and the assumption that all our observations are related to our positions, i.e. the position of the observers or Zuschauer, "dem Ort unseres Aufenthaltes".

[20] This method has been later rediscovered and renewed by Einstein as a fundamental principle of physics.

[21] Euler's theory on the origin of forces is different both from Newton's and Leibniz's explanations of the origin of forces (compare Ch. 5).

frame of concepts is built from time, space and motion. Euler continued the Newtonian tradition and discussed simultaneously the principles of mechanics for absolute motion and relative motion.

In 19th century the development of the theory of relativity was heralded by Mach's criticism of Newton's concept of absolute space and time.[22] However, Euler's contribution did not attract his attention despite the comprehensive treatise *Anleitung zur Naturlehre* was published in 1862. One of the reasons may be that Euler was considered in first respect as a mathematician despite his tremendous contribution to physics. The Chapters 10 and 11 of Euler's treatise *Anleitung zur Naturlehre* are devoted to that part of mechanics which is called nowadays relativistic mechanics. The treatise is aimed for investigation of *causes of changes* which are happen on bodies. Using the 18th century terminology we have to distinguish between *apparent* (*scheinbar* or *relative*) and *true* (*absolute)* motions.[23]

6.1 Euler on Absolute and Relative Motion

Euler's mechanics is in contrast to Leibniz's mechanics since Leibniz considered the velocity and the motion rather as a property of single bodies than as pure relations between bodies. In Newton's and Leibniz's mechanics coexist different schemes since relational and non-relational versions for the definition of forces are used simultaneously. This ambiguity was finally removed by Euler.

Euler's relational theory of mechanics is crowned by the introduction of observers who are associated with bodies and who are either resting or moving relatively to each other as far as the bodies perform these motions. The action of the observers is the measurement of distances between bodies and the comparison of results of their measurements [Euler E842, §§ 77–83]. Euler introduced a moving observer who is, moreover, not only aware of his own position (place) in the world, but is also taking part in the motion of bodies which are forming the system. The world consists of resting and moving *bodies* and resting and moving *spectators* (observers). In the introductory part of the *Mechanica* [Euler E015/016 §§ 1–96], the position of the observer is *indeterminate* relatively to a body. However, the observer is able to determine the relative positions of the bodies he is watching [Euler E015/016, §§ 77–93]. This point of view may be summarized in the statement where the observer is introduced by the statement "the body A to which *one* relates the motion of body B" or "the body A to which *we* relate the motion of body B".

[22] Mach did neither rely on Leibniz nor on Euler. In contrary, he condemned Leibniz for his metaphysics and criticized Euler for his digression in writing the *Letters to a German Princess* [Mach, Mechanik, p. 433].

[23] The analysis of Euler's theory will be mainly based on the treatises (i) *Mechanica*, 1736 [Euler E15/16], (ii) *Gedancken von den Elementen der Cörper*, 1746 [Euler E081], (iii) *Réflexions sur l'espace et le tems*, 1750 [Euler E149], (iv) *Décoverte d'un nouveau principe de Mécanique*, 1752 [Euler E177], (v) *Réflexions sur l'origine des forces*, 1752 [Euler E181], (vi) *Anleitung zur Naturlehre*, 1862 [Euler E842] and (vii) *Theoria motus corporum solidorum seu rigidorum*, 1765 [Euler E289], covering the three decades of Euler's work on mechanics from 1734 to 1765.

This indeterminate position will be later replaced with a position determinate with respect to *one of the bodies* the system consists of.[24] Furthermore, relative motion is an exclusive form of motion. Euler referred to his fundamental assumption on the exclusion of absolute motion that "we have only a relative imagination of the respective motion".

> 80. Quia omnis idea, quam de motu habemus est relativa (§ 7), haec quoque leges non sufficiunt ad cognoscendum, qualis sit cuiuspiam corporis motus absolutus. [Euler E015/016, § 80]

In §7 Euler stated that we cannot form determinate notions and ideas of the immensurable space and its boundaries.

> 7. Quoniam autem immensi illius spatii eiusque terminorum, quorum in datis definitionibus mentio est facta, nullam nobis certam formare possemus ideam, loco huius immensi spatii eiusque terminorum considerare solemus spatium finitum limitesque corporeos, ex quibus de corporum motu et quiete indicamus. [Euler E015/016, § 7]

Therefore, *instead of* using absolute space we make use of a finite space limited by corporeal boundaries. Euler continued

> 7. (...) Sic dicere solemus corpus, quo respectu horum limitum situm eundem conservat, quiescere, ido vero, quod situm eodem respectu mutat, moveri. [Euler E015/016, § 7]

and stated later, in complete agreement with the definition in *Mechanica*

> 7. (...) Quare si punctum O respectu istius corporis A eundem situm servet, quod fit, si ab omnibus eius punctis perpetuo aeque maneat remotum, tum punctum O respectu corporis A quiescere dicitur. [Euler E289, § 7]

Euler replaced the *indeterminate* position of the observer which was assumed in the preceding theories with a determinate position. Mostly, the observer was placed *outside* the investigated system of bodies and observed the motion inside the system. Hence, the whole system and the observer are resting relative to each another.[25] The attachment of observers to the bodies results in a drastic change of the models. Each of the observers is related to one of the bodies. The world as an object of investigation consists of bodies and observers who are measuring and constructing theories. The positions of the bodies are determined with respect to other bodies and, therefore, the positions of the observers are also defined with respect to other

[24] Here, it becomes obvious that these considerations are only complete if the system consists of a *finite* number of bodies and a *finite* number of observers. In the Zeno-Aristotle model, each of the drivers of the chariots has to be considered as an *observer* who is taking part in the motion of the "body", in that case the chariot, to which he is attached. Therefore, each of the driver-observers performed observations of the system which are different from the observations of all other driver-observers. Such a model had been developed by Leibniz [Leibniz, Monadology, § 57]. However, Leibniz did not consider quantitative differences related to the different perspectives which may result from relative motion.

[25] The ancient prototype of such models is the stadium discussed by Aristotle where three groups of chariots are moving relative to each another with different velocities [Aristotle, Physics]. The stadium is assumed to be the invariant frame of reference and observers are called only those who are sitting in the theatre, but not also those who are driving the chariots.

bodies and observers. Introducing *other* observers or onlookers who are attached to *other* bodies of the system, it is not only possible to consider the relative motion of bodies, but also *onlookers* who are resting or moving relatively to each other. Moreover, it turns out to be necessary to relate (i) the theories the different observers are using to each another and to compare (ii) the experimental results they obtained by measurement [Euler E842, §§ 77–83]. Here, except the consideration of light velocity as an upper limit for motions of bodies, we find all perquisites of the special theory of relativity later developed by Einstein.

6.2 Basic Models

In Ptolemy's model of the world the onlooker is placed at the earth which is resting as an immobile body in the centre of the universe. Relative motions have been discussed by Zeno who introduced instructive models which had been reconsidered by his followers. Aristotle discussed Zeno's paradoxes on motion. Descartes introduced relativity of motion beyond the frame where motion is considered as being paradoxical, but in the Cartesian version motion becomes *indeterminate*. A body can take different motions and velocities or, it is possible to assign to a body different velocities in dependence on the choice of the frame of reference, i.e. with respect to the other bodies whose state in independent of the state of the chosen body. Descartes explained the relativity of motion using the model system of a moving ship where a passenger is observing his environment on the ship and the shore the ship is leaving.

> For it is necessary, in order to determine this situation, to regard certain other bodies which we consider as immovable; and, according as we look to different bodies, we may see that the same thing at the same time does and does not change place. For example, when a vessel is being carried out to sea, a person sitting at the stern may be said to remain always in one place, if we look to the parts of the vessel, since with respect to these he preserves the same situation; and on the other hand, if regard be had to the neighbouring shores, the same person will seem to be perpetually changing place, seeing he is constantly receding from one shore and approaching another. [Descartes, Principles, II, § 13]

The Cartesian observer is always moving or resting at the same time, the description of state depends on the body of references which is either the ship or the shore.

> But motion, viz., local, for I can conceive no other kind of motion, and therefore I do not think we ought to suppose there is any other in nature, in the ordinary sense of the term, is nothing more than the action by which a body passes from one place to another. And just as we have remarked above that the same thing may be said to change and not to change place at the same time, so also we may say that the same thing is at the same time moved and not moved. Thus, for example, a person seated in a vessel which is setting sail, thinks he is in motion if he look to the shore that he has left, and consider it as fixed; but not if he regard the ship itself, among the parts of which he preserves always the same situation. [Descartes, Principles (Ross), II, XXIV]

This model had been also used later by Leibniz, Newton and Euler. However, Descartes did not distinguish between different kinds of motion performed by the bodies and the different behaviour of observers. The ancient prototypes of such models are well-known as Zeno's paradoxes[26] of (i) the arrow,[27] (ii) Achilles and the tortoise[28] and (iii) the stadium.

In the complete formulation of the problem, there are always two correlated motions each of them being unique from the point of view of each of the observers or, there is always a problem and the correlated inverse problem. This methodology had been successfully developed by Newton in the *Method of Fluxions* [Newton, Method of Fluxions] and in the Principia for the relation between phenomena and forces [Newton, Principia] (compare Chaps. 2 and 3). Following Euler, the representation of motion can also be formulated in terms of direct and inverse problems [Euler E065]. Nevertheless, both problems can be mostly separately formulated and solved because the methods for the formulation and treatment are quite different.[29] However, in *relativistic mechanics* the direct and the inverse problem cannot be separately formulated and solved since the essence of the theory is that "rest" can be assigned to any of the bodies, but the phenomena are the same, as it had been already pointed out by Leibniz. Moreover, Leibniz claimed that it is impossible to decide by the observation of several bodies to which of the bodies being in motion is to be given the status of absolute motion or absolute rest [Leibniz, Specimen, II (2)].

> Sic igitur habendum est, si corpora quotcunque sint in motu, ex phaenomenis non posse colligi in quo eorum sit motus absolutus determinatus vel quies, sed cuilibet ex iis assumpto posse attribui quietem ut tamen eadem phaenomena prodeant. [Leibniz, Specimen, II (2)]

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[26] Zeno's paradoxes have been analyzed from different points of view, but here we are only interested in the positions the observers are occupying *outside* or *inside* the observed systems.

[27] "The third is (...) that the flying arrow is at rest, which result follows from the assumption that time is composed of moments (...) he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now, the flying arrow is therefore motionless." [Aristotle, Physics, 239b.30]. Zeno abolishes motion, saying "What is in motion moves neither in the place it is nor in one in which it is not". (Diogenes Laertius *Lives of Famous Philosophers*, ix.72) [Stanford Encyclopedia] The observer is not sitting at the flying arrow, but is outside the system.

[28] "The [second] argument was called 'Achilles', accordingly, from the fact that Achilles was taken [as a character] in it, and the argument says that it is impossible for him to overtake the tortoise when pursuing it. For in fact it is necessary that what is to overtake [something], before overtaking [it], first reach the limit from which what is fleeing set forth. In [the time in] which what is pursuing arrives at this, what is fleeing will advance a certain interval, even if it is less than that which what is pursuing advanced (...). And in the time again in which what is pursuing will traverse this [interval] which what is fleeing advanced, in this time again what is fleeing will traverse some amount (...). And thus in every time in which what is pursuing will traverse the [interval] which what is fleeing, being slower, has already advanced, what is fleeing will also advance some amount." (Simplicius(b) *On Aristotle's Physics*, 1014.10) [Stanford Encyclopedia] Neither Achilles not the tortoise are considered as observers who are watching each other, but the only observers are outside the system and do not describe their own state as a simultaneously performed *twofold relative motion* with respect to Achilles and the tortoise.

[29] "In 1710, Johann Bernoulli pointed out that Newton had not proved Kepler's law of ellipses but only its converse and did so himself using the calculus, solving 'the general problem by reducing it to the same integral that is used to solve it today' (Park 1990, 416)." [http://www.sciencetimeline.net/1651.htm]

Hence, the choice of a frame of reference is arbitrary and is not determinate by the phenomena. Following Euler, in classical mechanics, a consistent theory of *relative motion* is compatible with the supposition of absolute space and time (compare Sect. 6.3). Hence, the direct and the inverse problem can be neither separately formulated nor solved. Both problems are automatically and simultaneously defined since there are *always* two observers who define *complementary* problems and experimental setups which have to be related to each other. Assuming any explicitly introduced or hidden absolute frame of reference, one part is truncated and the system is reduced to one body and one observer who is not related to, but separated from the observed resting or moving body.

6.2.1 The Model of Ship and Shore: The Observer in a Cabin on the Ship

An essential step toward a complete theory is the model of the ship moving relatively to a shore.[30] There is a passenger (first observer) on the ship who is moving together with the ship relatively to the shore and there is an angler (second observer) resting on a certain place on the shore. However, in most of the models both observers are not treated on an equal footing. The paradoxes of motion are manly caused by the incompleteness of the underlying models for relative motion. The different perspectives are not treated as equivalent description of the "world" made up of two bodies, the ship and the shore, and two observers, the passenger and the angler, who are watching the coast and waiting for a fish, respectively. The angler got the distinct impression that the fish is swimming to the place he is waiting for him. Hence, the fish is moving whereas the angler is resting. The passenger on the ship has the opposite feeling. He thought that the ship and he together with the ship are moving whereas the angler and with him the shore are resting (compare Sect. 6.2.4 for Châtelet's analysis [Châtelet, Institutions]). The passenger and the angler transformed their experience into a theory which contains at least a resting frame of reference. Motion is described as a change of position relatively to this frame. However, the simultaneous motion of the frame relatively to the observer is not completely, but only partially taken into account. Following Euler, the world is only made up of two bodies, the "ship" and the "shore" or the "angler" and the "fish", whereas water, wind, cabin, chairs, tables, fish-hook and angle are not taken into account as additional bodies although they may considered as extended things. An improvement of this model is due the introduction of a cabin on the ship where the passenger is sitting and is (i) either watching the environment or (ii) is separated from the rest of the ship as well as from the environment [Leibniz, Specimen, II (7)].[31]

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[30] Descartes [Descartes, Principia], Leibniz [Leibniz, Specimen].

[31] "(...) cum tamen certum sit, si qui in magna navi (causa si placet, vel certe ita constituta, ut externa a vectoribus notari nequeant) ferantur, navis autem magna licet celeritate, placide tamen

6.2.2 More than One Observer: The Stadium

The ancient prototype of a model for relative motion with more than one observer is well-known as Zeno's paradox of the stadium. The *stadium* discussed by Aristotle consists of three groups of chariots which are moving relative to each another with different velocities [Aristotle (Stanford), Physics, 239b 33]. The building is assumed to be the invariant frame of reference and observers are called only those who are sitting in the theatre, but not also those who are driving the chariots.

> The fourth argument is that concerning equal bodies [*AA*] which move alongside equal bodies in the stadium from opposite directions — the ones from the end of the stadium [*CC*], the others from the middle [*BB*] — at equal speeds, in which he thinks it follows that half the time is equal to its double (...). And it follows that the *C* has passed all the *A*s and the *B* half; so that the time is half (...). And at the same time it follows that the first *B* has passed all the *C*s. [Aristotle (Stanford), Physics, 239b 33] [Stanford Encyclopedia]

Hence, it might be very difficult to suppose that the drivers of the chariots are to be necessarily regarded as resting observers or, having in case of relative motion the same status of observers as the onlookers sitting in the circles of the stadium. Although Leibniz treated the different phenomena perceived by the observers, i.e. the drivers of the chariots and the onlooker in the circles, as equivalent [Leibniz, Specimen, II (2)], he concluded that there should be an internal difference between them which had to be traced back to forces. These forces are not related to the interaction since there is usually no interaction between drivers and onlooker, but to their "velocity" expressed in terms of "living force". Obviously, the drivers and the horses had to work whereas the onlookers are relaxing.

This different status of observers had been only removed by placing both observers in cabins where they are only able to look at each other, but not at anybody else. In the modern version such cabins are driven by electrical engines: Or, a passenger is sitting in the cabin of a train and looking at another passenger sitting in the cabin of another train. Now, if one of the two trains is leaving the railway station, none of the passengers can be sure that his train is leaving the railway station without making possible to observe a part of the railway station. This experimental setup had been theoretically modelled by Euler who investigates the equations of motions used by the passenger to describe motion or the change of motion.

6.2.3 Euler's Analytical Model of Relative Motion

The basic model had been introduced by Descartes who assumed that motion is the translation of a body A from the vicinity of a body B in the vicinity of a body C

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sive aequabiliter moveatur; ipsos nullum habituros principium discernendi (ex iis scilicet quae in navi contingunt) utrum navis quiescat an moveatur, etiamsi forte pila in navi ludatur, aliive motus exerceantur." [Leibniz, Specimen, II (7)] The model of the observers sitting in a cabin had been later used by Einstein to demonstrate the equivalence of acceleration and gravitation.

[Descartes, Principles]. Euler demonstrated that the basic relations of relative motion are already obtained by studying the *internal* principles of motion, i.e. *without* the consideration of forces. Instead of using bodies, Euler analyzed relative motion in the frame of differential equations establishing a complete abstract model of motion. Motion is represented by velocity and the body by a uniformly moving mass point. The change of motion is described by two observers who are moving relatively to each other with constant velocity. Motion is represented by the *analytical* expressions and the symmetry of the investigated relations. Hence, velocity is always doubly represented if it was previously defined as relative velocity since if the body A is moving relatively to B then the body B is necessarily also moving relatively to the body A. The confirmation of this symmetry might be cumbersome in many cases because of the asymmetry in the size of the two systems. However, comparing two mass points there is no difference in extension and even masses of different magnitude do not disturb the symmetry of relative motion as long as interactions are excluded, i.e. external principles of motion are being outside the scope.[32] As a consequence, the Leibnizian correlation between forces and motion [Leibniz, Specimen, II (2)] is completely removed and all theorems based directly or indirectly upon this assumptions are automatically abolished. The analytical model comprises all concepts being *necessary* and *sufficient* to discuss completely the properties of a system consisting of two uniformly moving bodies in term of the observations the involved observers made.

6.2.4 Motion as an Illusion. "Spitzfindigkeiten"

The disclosure of the symmetry of the model can be seriously hampered by the inclusion of bodies varying considerably in extension, shape and weight. Applying internal principles, theses differences can be taken for nothing although they might cause strong sensual impressions. This misleading drawback may be already convincingly demonstrated by the model of ship, shore and passenger discussed by Descartes [Descartes, Principles]. The ship has compared to the passenger an as tremendous mass as the shore has in comparison to the ship. The problem to handle properly these disparities in mass and size will be demonstrated by the writings of Euler's contemporaries Châtelet (1706–1749) and Kästner (1719–1800).

In Châtelet's approach essential Leibnizian assumptions on the relation between rest and motion are preserved, especially the assumption that a motion can only be generated by a motion [Leibniz, Specimen]. As a result, Châtelet *overestimates* finally the state of motion and *underestimates* the state of rest.

[32] Although the procedure is appropriate to demonstrate symmetries, Euler's contemporaries become unhappy with this abstract approach to real bodies and real motion. Lichtenberg responded ironically to the attempt to demonstrate the laws of motion analytically. "In einem Buch von der Tanz-Kunst könnte erstlich die Kreatur als ein Punkt betrachtet werden, die noch keinen Hintern, noch keine rechte und linke Hand hat, so wie Euler die Mechanik abhandelt." [Lichtenberg, Heft B 28]

Both the states are never treated on an equal footing. Despite the heading of the Chap. XI entitled *De Mouvement, & du Repos en général, & du Mouvement simple* [Châtelet, Institutions], Châtelet did not only treat rest and motion separately, but also essentially different. This will be demonstrated for the case of relative motion which is discussed in the famous Cartesian model of a ship travelling along a shore while a passenger, sitting or walking on the deck of the ship, is watching the shore and the desk positioned on the deck of the ship.

> Celui qui est dans le Vaisseau & qui croit que la pierre a marché d'Orient en Occident, attribue à la pierre le mouvement qui n'appartient qu'an Vaisseau; & il est trompé par ses sens de la même manière que nos sommes, quand nous croyons que le rivage que nous quittons s'ensuit, quoique ce soit le Vaisseau qui nous porte qui s'en éloigne, car nous jugeons les objets en repos, quand leurs images occupent toujours les mêmes points sur notre rétine. [Châtelet, Institutions, § 219]

Châtelet claimed that we are cheated by our senses if we believe that the shore is escaping from us. The *phenomenological* part of the problem is rightfully described as far as Châtelet considered an observer BOBShip who is *resting relatively* to the ship. However, keeping the observer BOBShip at rest an observer BOBShore who is sitting on the shore is simultaneously moving away from the position of BOBShip in the same time and by the same distance as BOBShip is moving away from BOBShore. There is no difference to the case where the ship is resting in the port and is *not* moving relatively to the shore.

Here, Châtelet assumed the Leibnizian relational model of spatial positions where a privileged group of bodies is assumed which is considered as a system of reference, i.e. the observer is always attached to that group bodies. The other bodies in this world which are moving relatively to that privileged group are not equipped with any of observers.

Châtelet runs into trouble since she based her considerations on the Leibnizian *relational* definition of space which turned out to be different from the *consequent* relative or *relativistic* definition of spatial distances. However, it can be demonstrated that Leibniz invented an incomplete representation of relativity since he implicitly assumed the same preferential state of the observer which has been assigned to God by Newton. The Leibnizian observer does not move simultaneously as any of the body is moving, but he watches the bodies which are changing their position relatively to him without taking the complementary position obtained by an own motion relatively to resting bodies.

Leibniz introduced an operational definition of relative translation in terms of *geometrical* relations which may be easily transferred to relative motion.

> 47. When it happened that one of those co-existent things changes its relation to a multitude of others, which do not change their relation among themselves; (...). And, to give a kind of a definition: *place* is that, which we say is the same to A and, to B, when the relation of the co-existence of B, with C, E, F, G etc. agrees perfectly with the relation of the co-existence, which A had with the same C, E, F, G, etc. (...) Lastly, *space* is that, which results from places taken together. [Leibniz Clarke])

Descartes reduced the multitude of coexisting things to two bodies AB and CD. The relations between AB and CD are defined *analytically* without reference to a further body. The only quantities are two distances, (i) the distance measured between AB and CD taken from body AB and (ii) the distance measured between CD and AB taken from body CD as point of reference. All the bodies and all the possible observers are *equivalent internal* parts of the system.

In Leibniz's model the group of bodies C, D, E, F is in an invariant position whereas the body A is changing his position by translation relatively to the group. Finally, a body B is considered as a body "newly come" which changed its position also to the group which it assumed to be at rest [Leibniz Clarke]. Therefore, completing the world with observers (nothing else than Leibnizian monads), Leibniz assumed that the group C, D, E, F is associated with an observer BOB-CDEF whereas the bodies A and N are no associated to observers: CDEF & BOB-CDEF, A & non BOB-A, B & non-BOB-B. Leibniz assumed the bird's eye view of an observer who is neither associated with the group C, D, E, F nor with A nor with B. This observer has to decide which of the body is resting and which of the bodies is moving relatively to this resting body [Leibniz, Specimen, II (2)]. The rest is assigned to any of the bodies the system consists of.

> Sic igitur habendum est, si corpora quotcunque sint in motu, (...) sed cuilibet ex iis assump to posse attribui quietem ut tamen eadem phaenomena prodeant. [Leibniz, Specimen, II (2)]

In Leibniz's model the implicit assumption of the non-equivalence of rest and motion is incorporated. Nothing is changed in the procedure where the relative positions are determined by the *observers* who are associated with the bodies. The only question to answer is: Are the relative positions preserved or changed? In the first case, all bodies or a certain subgroup are in the state of *relative rest*, in the second sace, all the bodies which relative positions are changed are in motion, i.e. in *relative motion*. (compare Sect. 4.4).

The conclusion is that Leibniz considered the change of position [Leibniz, Specimen, II (2)] purely geometrically, i.e. independently of (or separated from) forces. From Châtelet's interpretation we learn that even in the case of purely phenomenological description of motion we are only prevented from trouble if we assume Leibnizian monads as parts of a world which is free of forces. The only activity of the monads is the creation of a perception depending on the perspective.

> 57. And as the same town, looked at from various sides, appears quite different and becomes as it were numerous in aspects [perspectivement]; even so, as a result of the infinite number of simple substances, it is as if there were so many different universes, which, nevertheless are nothing but aspects [perspectives] of a single universe, according to the special point of view of each Monad. [Leibniz, Monadology, § 57]

Although Leibniz assumed a relational theory of time and space, he did not introduce a *complete relational theory of motion* since he did not accept the relative motion and, consequently, the relative velocity of two bodies as an *invariant* item of a system of bodies. The reason is that Leibniz did not introduce velocity as a *relation* between bodies and refused to assign rest and motion, i.e. a certain value of velocity, to each of the bodies of the system. This procedure results unavoidably in the question of the position of the body in the *space* which has been answered by

Newton by the introduction of *absolute* space. Euler removed this question by the consequent relational theory of motion [Euler E842, § 77].

Châtelet compared the observation of the motion by different observers who are placed on the ship (observer A(ship)) and on the shore (observer B(shore)). In agreement with the previous definitions the shore is in absolute rest since the parts of the shore are not moving relatively to each other.

> 218. (...) cette pierre paroîtra à ceux qui sont dans le Vaisseau [A(ship)] avoir un mouvement relatif propre (besondere Bewegung nebst anderen Dingen), dans le sens dans lequel on l'a jettée; mais ceux qui sont sur le rivage [B(shore)] la verront dans un repos absolu, par rapport à sa direction horisontale, & ce repos est son état réel. [Châtelet, Institutions, § 218]

Later, Einstein discussed the model consisting of a passenger A who is sitting in the moving train, a passenger B who is waiting at the railway station and, additionally, a raven who is flying in a direction parallel to the rails [Einstein, Allgemeine Relat]. Furthermore, A *third* observer is viewing the whole arrangement of rails, train, railway station and raven who is related neither to the rails nor to the train nor to the railway station nor to flying raven, i.e. whose position is indeterminate. This third man is either God or the author of the theory. Intuitively, he is associated with the biggest of the objects which are forming the system, i.e. the railway station and the earth. Both the constituents, the third man and the biggest body are resting, i.e. they are in the state of in relative rest.

Châtelet assumed rightfully a state of absolute rest which is defined relatively to an observer. The observer is always associated with bodies which are resting relatively to him.[33]

In 1766, Kästner summarized the state of art as far as relative motion is based on the Leibnizian model [Kästner, Anfangsgründe]. As Châtelet assumed for the analysis of the motion of the ship relatively to the shore, Kästner assumed the same basic configuration, a walker and a tree instead of the ship and the shore, and demonstrated unintentionally that the parts of the system and the observers are *not* equivalent internal parts of the system.

> Wenn wir einen Menschen in der Ferne auf dem Felde sehen, und nicht eigentlich erkennen können, ob er fortgeht oder stille steht, so werden wir acht geben, ob er seine Lage gegen unbewegliche Gegenstände, einen Baum, einen Hügel u.s.f. ändert oder nicht. Wenn also der Mensch stille stünde, und der Baum oder der Hügel sich von ihm entfernte oder näherte, würde wohl dieses heissen der Mensch veränderte seinen Ort, und bewege sich also? Mit dieser Spitzfindigkeit hat man noch im vorigen Jahrhundert die verwirrt, die die corpenicanische Weltordnung verketzerten. (...) Wer nicht Lust am Zanken hat, wird leicht unterscheiden, ob ein Körper seinen Ort verändert, oder ob andere Körper ihren Ort um ihn verändern. [Kästner, Anfangsgründe, I. Cap. § 1 and 2]

[33] Using contemporary terminology, the observer A is measuring in a lab A which is *not moving* relatively to him. This procedure is completed by measurements which are performed for bodies (and other labs, called lab B, e.g. a clock B which is moving relatively to the clock A the observer A is using) which are moving relatively to lab A.

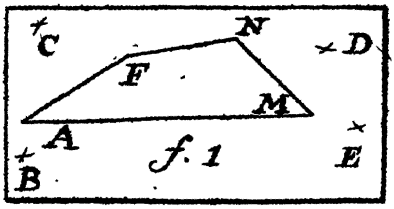

Fig. 6.1 Kästner's model for relative motion [Kästner, Anfangsgründe, Fig. 1]

Following in goal and spirit Leibniz's original version, Kästner discussed a system of bodies whose relative motions are described by the change of the places or "Gegenden" (see Fig. 6.1).[34]

> 3. Die Körper, B; C; D; E; 1. Fig. vermittels der man den Ort des Körpers A bestimmt, von dessen Bewegung die Rede ist, mögen seyn was sie wollen, und liegen wie sie wollen. Wenn der Körper von A nach F geht, so wird man seine Bewegung daran erkennen, dass er sich von der Gegend wo B;E; liegen entfernt, nach der wo C;D; befindlich sind zugeht. [Kästner, Anfangsgründe, § 3]

Leibniz constructed a very complicated system of bodies to explain at first relative motion and at second to exclude the appearance of an empty place or vacuum which may be generated by a body leaving its place by translation or motion ("newly come").[35] Hence, in Leibniz's model, the whole universe is involved if one of the bodies changes its place since the previous place of the "newly coming" body has to also simultaneously be reoccupied by another body also "newly coming" and so on in infinity. Obviously, the description will be never complete as long as the changing of places in the whole universe is known (Leibniz established a many-body theory where any body-body relation is only approximately valid). This is impossible to attain by a creature. Hence, Châtelet and Kästner truncated the procedure by referring to senses. Euler established a system of notion which is independent of recourse like that by resolving all possible interactions in body-body interactions.

Euler reduced the number of bodies in Leibniz's model and Kästner's version of the model exactly to *two bodies* since all peculiarities of relative motion can be explained by the relations between these two objects (two body model) without any additional "hidden" components.[36] Furthermore, Euler supposed empty space or vacuum as *common background* for both the bodies. Hence, he had no to bother

[34] For the terminology of "Gegenden" (directions) compare Euler [Euler E842, §§ 57 and 77–83].

[35] "(...) and that another thing, newly come, acquires the same relation to the others, as the former had; we then say, it is come into the place of the former; and this change, we call a motion in that body, where it is the immediate cause of the change." [Leibniz, Clarke (Alexander) 5th Paper, pp. 69–70]

[36] "Similiter si duo corpora A et B sibi concurrant, manente motu eodem ipsius B, continue imminui velocitatem ipsius A, donec ea omnino evanescat seu nulla fiat ipsius A celeritas, licebit casum hunc cum casu motus ipsius B una ratiocinatione complecti." [Gerhardt, Historia] "Similarly, if two bodies are in motion at the same time, and it is assumed that while the motion of B remains the same, the velocity of A is continually diminished until it vanishes altogether, or the

what will happen with the empty place the body left by changing its position. The body-body interaction can be exactly and completely described without any approximations. Euler assumed that any body-body interaction between bodies A and B is *independent* of the interaction of these bodies with other bodies C, D, E, F &c. Then, mechanics is the theory to study this two body problem appearing in (i) relative rest or motion and (ii) interaction of bodies. The interactions in the world are *not reduced* to body-body interactions, but are composed of these elementary events which cannot be further decomposed into other events.

6.3 Euler's Relational Theory of Motion

Euler's relational theory of motion is based on the distinction between *internal* and *external* principles or *force independent* and *force dependent* changes of the position of bodies. The position of a body is always determinate by the relation to other bodies. As it had been demonstrated, Newton's concept of absolute time and space is compatible with the introduction of relative motion and the rejection of absolute motion [Euler E015/016]. The persistence of the idea of absolute time and space is guaranteed by the supposition that the "increment of velocity is independent of the velocity" [Euler E015/016, § 131]. Then, interpreting a statement of Euler on the validity of the basic equation of motion [Euler E015/016, § 152], Newton's theory is not "only valid, but necessarily valid" [Euler E141, §§ 20 and 21] since the increment of the variable "time" turned out to be in magnitude independent of the motion of bodies. Consequently, although the increment of time is indeterminate in magnitude, it is known to be *constant* for all motions performed by bodies or $o = \text{const}$. Newton's time and increment of time are *universal* quantities or *universal* parameters.

6.3.1 The Analysis of Basic Concepts

Euler's theory was published in a complete and systematic representation in *Anleitung zur Naturlehre* [Euler E842] in 1862. In the treatise Euler developed the most complete version of the relation between bodies and observers. The basic statements of the *Mechanica* [Euler E015/016, §§ 7 and 97] are elaborated in all aspects of the problem. The theory is completed by the introduction of more than one observer who are comparing their theories for describing the action of forces.

The relation between absolute and relative motion is discussed from the same point of view which is known from the *Mechanica* and the *Theoria*, but the decision

speed of A becomes zero; it will, be permissible to include this case with the case of motion of B under one general reasoning." [Leibniz (Child), p. 147] Hence, Leibniz discussed the motion of the bodies A and B with respect to a "hidden body" C. Body B maintains the motion with respect to C whereas A diminished its motion with respect to the same body C.

in favour of relative motion is more rigorous and pronounced. Using the contemporary terminology, the *absolute* motion is called *wahre* (true) Bewegung and the *relative* motion is called *scheinbare* (apparent) Bewegung. The true (wahre, absolute) motion is mentioned for the first time for the motion of a body at a plane [Euler E842, § 72]. The motion is described by the components which are related to two straight lines oriented perpendicular to each other. Obviously, this motion is not related to the whole absolute space as it was usually assumed by Newton, but to a certain fixed subspace and a finite environment of that subspace, the finite distance the body is moving at the plane. The same consideration is transferred to the motion on curved lines in the space where the motion is related to three planes oriented perpendicular to each other [Euler E842, §§ 73–74]. These planes define a certain special point by their intersection which is distinguished from all other points of the space. In the paragraphs at the end of Chap. 9 of the *Anleitung* [Euler E842], Euler discussed the independence of the *path integral* for conservative forces of the path. This quantity, called *Wirksamkeit*, is related to the living forces [Euler E842, §§ 74–76].

The treatise *Anleitung zur Naturlehre* [Euler E842] is organized as follows. The titles of the Chapters listed here may be red as a short version of Euler's program for mechanics including a summary of the progress Euler made since 1736. Chapter 10 entitled *On apparent motion* is of special interest for the theory of relative motion.

1. Von der Naturlehre überhaupt (on the science of nature in general)
2. Von der Ausdehnung (on extension)
3. Von der Beweglichkeit (on mobility)
4. Von der Standhaftigkeit (on steadfastness)
5. Von der Undurchdringlichkeit (on impenetrability)
6. partially missing
7. Von der Wirkung der Kräfte auf die Geschwindigkeit der Körper (on the effect of forces on the velocity of bodies)
8. Von der Wirkung der Kräfte auf die Richtung der Körper (on the effect of forces on the direction of bodies)
9. Bestimmung der Bewegung eines Körper, welcher von Kräften getrieben wird (on the determination of motion of a body which is driven by forces)
10. Von der scheinbaren Bewegung (on the apparent motion)
11. Allgemeine Grundregeln der Naturlehre (general rules for the science of nature)
12. Von dem Unterschied der Körper in Vergleichung ihrer Ausdehnung und Standhaftigkeit (on the difference between bodies concerning the comparison of their extension and steadfastness)
13. Von den besonderen Eigenschaften der groben und subtilen Materie (on the special properties of coarse and subtile matter)
14. Von dem Äther (on the ether)
15. Von der Flüssigkeit (on fluids)
16. Von den verschiedenen Gattungen der Körper (on the different categories of bodies)
17. Erklärung der Festigkeit der Körper (explanation of the strength of bodies)

18. Von der Zusammendrückung und Federkraft der Körper (on the compression and the spring force)
19. Von der Schwere (on gravity)
20. Von den Gesetzen des Gleichgewichtes flüssiger Materie (on the laws of equilibrium of fluid matter)
21. Von den Gesetzen der Bewegung flüssiger Materien (on the laws of motion of fluid matter)

In the Chaps. 1 to 6 Euler demonstrated his axiomatic foundation of mechanics which is based on the notion of *impenetrability*. The subject of the *Naturlehre* or *science of nature* is the investigation of the changes which are happen to the bodies. [Euler E842, § 1]. For every occurring change of the states of bodies one has to demonstrate that even that change of the states had to have arise from the impenetrability [Euler E842, § 50]. The indispensable properties of all bodies are *extension, mobility* and *steadfastness* (compare [Euler E842, Chaps. 2, 3, 4]). However, Euler claimed that these properties follow necessarily from the *impenetrability* [Euler E842, Chap. 5]. The basic assumption is that every body is occupying a special place and it is impossible, that two bodies may simultaneously stay at the same place [Euler E842, § 35]. Therefore, it is excluded that the bodies penetrate each other. Euler stated that the notion of body includes necessarily the impenetrability and, moreover, claimed that Descartes also had believed that impenetrability is connected with extension despite he asserted res extensa sive corpus [Euler E842, § 35].

The further development of the theory is straightforwardly. All essential feature are related to rest and motion being the only states the body is preserving due to internal principles of motion [Euler E842, Chaps. 4 and 5]. The change of the states, either the change of rest or the change of motion, is due to forces which are generated by the bodies to avoid penetration. The forces appear due to impenetrability of at least two interacting bodies, the result is a *mutual* change of the states of the bodies involved in the interaction. It is impossible that one body can induce a change of his own state. The bodies are *not* distinguished by their *impenetrability*, but by their *inertia* which can be expressed in terms of their masses [Euler E842, Chaps. 6, 7, 8, 9]. In these Chapters Euler demonstrates the application of his method, called *Auflösungskunst*, which allows for a *complete* description of motion by decomposition of all motions into one, two or three components defined with respect to one, two or three straight lines or directions oriented perpendicularly to one, two or three planes. Euler analyzed the properties of conservative forces and defined the potential energy of a system, called *Wirksamkeit*, which is related to Maupertuis principle of least action.

In Chap. 10 of the *Anleitung* [Euler E842] Euler introduced explicitly *observers* or *Zuschauer* who had been waited behind the scene for their entry. Without addressing any introductory remarks to the reader, Euler pushed everybody onto the market of observations to struggle for and to come to an agreement concerning information about body's rest and motion. It is essential to agree upon information because any general law cannot be obtained without checking the different collections of data to confirm the common ruling principles behind them. These ruling principles are assumed to be either as necessary or as contingent. The necessary principles are indispensable to distinguish bodies from other things which are not bodies [Euler E842, §§ 1–6].

6.3.2 The Introduction of Observers, Zuschauer

Usually, the physicists are accustomed to the concept of an observer related theory, i.e. the role of the observation and the decisive role of the actions of observers, which had been clearly defined in relativistic physics [Einstein, Bewegte] for the first time and finally acknowledged in quantum mechanics in the 20th century [Bohr, Nobel], [Heisenberg 1925], [Heisenberg, Quantentheorie]. However, over 200 year ago, Euler did the same step when analyzing the basic principles of mechanics. Very early in 1734, Euler introduced the observer called *spectator* in the theory [Euler E015/016, § 97] and devoted one paragraph to him. Now, in the *Anleitung*, the observer is called *Zuschauer* (the book is written in German) and a whole Chapter is devoted to the discussion of his place in the frame of notions. In Chap. 10, entitled *Von der scheinbaren Bewegung* (On apparent motion), Euler declared without addressing some introductory remarks to reader that the *relative* motion is related to an *onlooker*[37] and is determined by the direction where the body is appearing for the onlooker and by the distance measured from his position.

> 77. Die scheinbare Bewegung bezieht sich auf einen Zuschauer und wird durch zwei Stücke bestimmt, erstlich aus der Gegend, nach welcher dem Zuschauer ein Körper erscheint, und hernach aus der Entfernung desselben vom Zuschauer. Dieses ist der scheinbare Ort des Körper und aus der Veränderung desselben wird die scheinbare Bewegung geschätzt. [Euler E842, § 77]

In the following text, Euler compared the estimations of two observers and claimed that their results are equivalent if they have beforehand defined equal directions by parallel lines. Moreover, the origins of the straight lines are assumed to be the eyes of the observers.

In the treatise, nothing is said about absolute or true (wahre Bewegung) motion except in relation to apparent motion. This type of motion does not exist as an autonomous theoretical subject. Therefore, we can conclude that Euler observed the different procedure in the *Theoria* only for sake of common reasons resulting from the needs and conditions of that time where people were accustomed to absolute motion. In the comment to the paragraph 77, Euler emphasized the relevance of *relative* or *apparent* motion which exclusively results from the exceptional role of the *observer* or *Zuschauer*. The motion is not only defined with respect to a chosen body but even preferentially with respect to the position of the observer.[38]

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[37] Remember that previously, in the *Mechanica*, the relative or respective motion had been defined with respect the fixed boundaries of space or relatively to other bodies.

[38] The reason is that motion is not explained by the properties of bodies, but by the relations between bodies which do not modify the properties of the bodies. "Ad quod ergo praedicamentarum genus referri debeant quies et motus, Philosophi viderint; qualitates certe minime vocari

> 77. (...) Von dieser (der scheinbaren) Bewegung ist um so viel nöthiger hier zu handeln, da wir uns in der Welt keinen anderen Begriff als von der scheinbaren Bewegung machen können; denn wir können die Oerter der Körper nicht anders als nach dem Orte unseres Aufenthaltes schätzen. [Euler E842, § 77][39]

If a resting observer is not changing his position and the directions are estimated by the same straight lines then the apparent motion is not distinguished from the true motion [Euler E842 § 78]. Beside the *position* of an observer, Euler introduced the estimation of the directions which are represented by straight lines.

6.3.3 The Priority of Relative Motion

Euler stressed the exclusive role of relative motion making a decision in favour of Leibniz and constituting a breach with the concept of absolute motion. Euler anticipated essential parts of Mach's criticism a century before Mach laying down a new ranking between time, space and motion. Obviously, relative motion is favoured by the introduction of the observers into the theory. Additionally, the reader can easily cast in doubt with more or less passion also absolute space and time. However, Newton's concept of relative quantities is confirmed and preserved. This radical change had been either overlooked or underestimated by the followers including Mach since Euler developed in all publications both the absolute and the relative approaches in parallel and almost everywhere on an equal footing.

Euler analyzed different experimental setups, (i) one observer who is changing his position (compare §§ 78 and 79), (ii) one observer who is changing his position and the direction of his observation, (iii) *two observers* who are comparing their observations (compare § 77) and (iv) two observers who perform observations from different places but preserving or establishing parallel directions. Hence, the traditional scheme of the *motion of bodies* is extended by the inclusion of the *motion of the observers*.

> 78. (...) war es doch nöthig, hier wohl bemerkt zu werden, damit man um so viel leichter den Unterschied zwischen der wahren und der scheinbaren Bewegung einsehen möge, wenn der Zuschauer seine Stelle verändert. [Euler E842, §§ 77–80]

In all cases Euler distinguished between uniform and non-uniform motion. The uniform motion is defined for *parallel straight* lines and *constant* velocities. The whole system is formed by the observers and bodies being *coexisting parts* of the system. Therefore, the relative velocity is defined for two observers who are

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possunt, nihil autem prohibit has res inter relationes numerare, quandoquidem, utcunque eadem res cum aliis aliisque obiectis comparetur, eius indoles interna nullam mutationem subit" [Euler E289, § 17].

[39] "77. *The apparent motion is related to an observer*, (...). In any case and for principal reasons, we are in great need to consider apparent motion since we cannot get another idea as of apparent motion if we are in the world, since we are not able to estimate the positions of the body in another way except according to the place of our stay." [Euler E842, § 77]

moving relatively to each other. The numerical value v_{rel} is an invariant quantity since it can be defined only with respect to both observers or to both the bodies which are the constituting parts of the system. Here in the same treatise, Euler completed the program for the *investigation of motion in the presence of forces* by a program for the investigation of relative motion of bodies in the *absence* of forces. Time and space are treated on an equal footing [Euler E149]. This procedure had been later continued by Lagrange and, in the 20th century, by Einstein and Minkowski.

6.3.4 The Invariance of the Equation of Motion

In the next step, the conditions for the validity of the equations of motion $dv_i = (K_{ik}/m_i)dt$ will be analyzed if different observers make use of them in the presence of forces. Following Euler, the world is made of two observers and two bodies. The observers and the bodies are moving uniformly relative to each other. Observers and bodies are labelled by indices $i,k = 1,2$. The only difference between them is (i) due the different *masses* m_1 and m_2 of the bodies[40] to whom they are partnered as *massless* onlookers and, (ii) due the *different perspectives* and, consequently, *different data* they obtain from their views at their common world. As it had been discussed in Sect. 6.2.4, the commonly accepted model of relative motion was quite different. Usually, only one observer was believed to be sufficient who, moreover, was settled at a privileged place *outside* the system (see Fig. 6.1).

Euler included the observers into the system and defined simultaneously the subject matter of investigation, i.e. the observation of the world from their own system of reference whose properties depend on the properties of bodies. Hence, the possible states are either (i) the state of relative rest or (ii) the state of relative motion. Furthermore, the observers are able to distinguish between uniform and non-uniform motions by monitoring the magnitude of relative velocities.[41] In the first case of uniform motion, no forces are needed to describe the persistent state of both bodies. In the second case of non-uniform motion or a change of the state of bodies, additional terms representing forces are to be added. Euler removed all parts of Kästner's model (compare Fig. 6.1) including the "hidden" onlooker outside the "world" except the straight lines AF and MN. The "hidden" observer *outside* the world is replaced with *two* observers BOB-AF and BOB-MN *inside* the world.

> 78. Dieser Zuschauer irrt also nicht, wo er nach den Regeln der Bewegung glaubt, dass entweder zu einer Bewegung Kräfte erforderlich sind oder nicht; denn diejenigen Körper, die ihm scheinen in ihrem Zustand zu verharren, verharren darin auch wirklich; und diejenigen Veränderungen, die ihm scheinen vorzugehen, die gehen auch wirklich vor, und werden

[40] The masses are the *inert* masses of the *non-interacting* bodies. The heavy masses are not considered in Euler's theory of relative motion.

[41] As it had been discussed above, there are always two experimentally detected values of relative velocity since there are always two observers.

zu deren Hervorbringung eben derjenigen Kräfte erfordert, welche im Vorigen bestimmt worden; (...). [Euler E842, § 78]

Now, all advantages of the analytical representation of motion are exposed to be viewed at once. The onlooker can be settled into a two-dimensional world which is formed by a plane. All events outside the world beyond the plane are not of interest since, due to the conservation of momentum, the motion of the bodies is confined to the plane (compare Chap. 4). A further reduction to a one-dimensional world formed by a straight line is straightforwardly. Analytically, only one equation of motion for each observer is to be taken into account. For uniform motion, it follows $dv_1 = 0$ and $dv_2 = 0$ for the preservation of the relative velocity. Any other case, either (a) $dv_1 \neq 0$ and $dv_2 = 0$ or (b) $dv_1 = 0$ and $dv_2 \neq 0$, is excluded. The consistency of the model will be confirmed by studying analytically (i) the effect of a finite constant velocity added to one of the bodies, $v_1^{\text{after}} = v_1^{\text{before}} + \alpha$ and (ii) the effect of forces which results in change of the velocities $dv_1 \neq 0$ and $dv_2 \neq 0$.

Let the two observers be denoted by BOB1 = BOB and BOB2 = BOB$'$. Both are using the relation $dt = dt'$ for the time intervals while observing the same body. The body with the mass $m = \text{const}$ is moving under the influence of a force described by its components P, Q, R (here, only one direction is considered, $P = K$). Then, the observers obtain the relation $ddx = ddx'$ if they agree to the condition formulated in § 77 of the *Anleitung*. Nowadays, the result is represented in terms of the Galileo transformation, i.e.

$$x' = x - v_{\text{rel}} \cdot t \qquad \text{and} \qquad t' = t. \tag{6.1}$$

Therefore, in case of relative motion, Euler analyzed the invariance of the equation of motion for Galileo transformations assuming the Newtonian simultaneity, i.e. preserving only the absolute time, but rejecting Newton's concept of absolute motion and, straightforwardly, challenged indirectly Newton's absolute space. Furthermore, Euler considered always the actions of two observers including their mutual relations in the frame of relative motion [Euler E842, §§ 77 and 79].

78. Wenn der Zuschauer an demselben Ort immer unbeweglich verharret und die Gegenden durch einerlei Linien schätzet, so wird die scheinbare Bewegung eines jeglichen Körpers von der wahren Bewegung nicht zu unterscheiden sein und also kommt das Urtheil dieses Zuschauers von der Bewegung aller Körper mit der Wahrheit überein. [Euler E842, § 78]

Then, Euler analyzed that a permanent change of the directions, e.g. a rotating frame of reference is not appropriate for the investigation.

79. Wenn der Zuschauer nicht nur seinen Ort, sondern auch seinen Begriff von den Gegenden beständig verändert, der andere aber, wenn er die Gegenden richtig schätzt und in seinen verschiedenen Stellungen diejenigen Gegenden für einerlei hält, welche durch gleichlaufende Linien bestimmt werden. [Euler E842, § 79]

The observer has to prepare his own motion such that he is moving uniformly along a straight line. Furthermore, the observer has to *agree* with *other* observers in the determination of the *same directions* by parallel lines if he intended to compare his observations with the observations of other observers.

> 77. (...) und wenn wir zwei Zuschauer setzen, so sind ihnen diejenigen Gegenden einerlei, welche durch Linien, so einander gleichlaufend oder parallel sind, bestimmt. [Euler E842, § 77]

If the observer is moving uniformly along the same direction (Gegend) then all bodies which are either resting or also moving along a straight line are appearing for him to persist in the same state.

> 80. Wenn der Zuschauer gleichgeschwind in einer geraden Linie fortrücket, (...). [Euler E842, § 80]

No force is necessary for the maintenance of such a motion of bodies. The observer can conclude that no force is acting upon the body. Then, the observer can estimate the action of forces from the change of uniform motion of the bodies. For the relation between the velocities Euler discussed the differences between the velocity of the body u and the velocity of the frame of reference a, i.e. $u - a$, where the frame of reference is established by a *plane* perpendicular to the direction of motion. The distances are estimated from the intersection point of the straight line with the plane (compare also [Euler E177]). The differential du is the same as before since the motion of the plane is uniformly, i.e. $da = 0$. The velocity a does not enter in the equation of motion for the change of velocity caused by forces.

$$dv = \frac{K}{m}dt \qquad d(v-a) = dv = \frac{K}{m}dt \tag{6.2}$$

The conclusion is that it is not necessary to consider additional forces. The forces for the apparent (or relative) motion are the same as the forces for the explanation of the true (or absolute) motion. The same result is obtained if the true motion is not due to forces. Then, the apparent motion is also independent of any forces [Euler E842, § 81]. This conclusion is readily confirmed by the analytical representation. Obviously, for the force-free case $K = 0$, from Eq. (6.2) it follows (i) $dv = 0$ and (ii) $d(v-a) = 0$.

> 81. Wenn der Zuschauer gleichgeschwind in einer geraden Linie fortrücket und die Gegenden richtig, das ist nach gleichlaufenden Linien schätzet, so werden zur Unterhaltung der scheinbaren Bewegung, (...), eben diejenigen Kräfte erfordert, als zur Unterhaltung der wahren Bewegung. [Euler E842, § 81]

The situation is considerably modified if the observer or the frame of reference, i.e. the plane where the observer is positioned, are *not* moving *uniformly*. Although preserving the same direction, the result is different from the case of uniform motion since additionally forces are needed.

> 82. Wenn der Zuschauer sich nicht gleichförmig in einer geraden Linie bewegt, dennoch aber die Gegenden richtig schätzet, (...), so werden ausser den Kräften, welche wirklich auf dieselben [die Körper] wirken, solche Kräfte erfordert, welche in einem jeden Körper alle Augenblicke eben die Veränderung hervorbringen, welche in dem Ort des Zuschauers vorgeht, aber nach der umgekehrten Richtung. [Euler E842, § 82]

The observer detects these forces by the change of the state of the body. The body behaves differently compared to state of uniform motion. Nowadays these forces are called *fictitious forces* (*Scheinkräfte*) or *Trägheitskräfte*.

6.4 Mach, Einstein and Minkowski

In the 18th century, Descartes' and Newton's mechanics had been treated as the basis for alternative models, especially for the explanation and interpretation of the universal attraction between all bodies [Nick, Gegenmodelle].[42] In the 19th century, Newton's principles had been accepted in the physics community. In 1883, distinguishing between the conceptual basis and the "deductive, formal and mathematical development" of mechanics,[43] Mach initiated a criticism of Newton's concepts of absolute space and time [Mach, Mechanik].[44] In the last decades of 19th century, the properties of light had been carefully studied by various experimental methods. The results seemed to be incompatible with the principle of relativity since the light was seemingly playing the same role which had been assigned to Newton absolute time and space in former times. Both quantities are independent of the dynamical state of bodies.

[42] Alternative models to Newton's theory of gravitation had been developed by Leibniz [Leibniz, Tentamen de motuum coelestium causis, 1698], Johann Bernoulli [Bernoulli, La Nouvelle Physique Céleste, 1734] and Daniel Bernoulli [Recherches physiques et astronomiques, 1734]. In the 18th century, the main objection was made againsts Newton's theory of interaction at distance (compare Nick, *Kontinentale Gegenmodelle zu Newtons Gravitationstheorie* (Nick, Gegenmodelle).

[43] Following Mach, the same relation between physical principles and their formal representation can be observed in Maxwell's mathematical formulation of Faraday's ideas. "In größtem Ansehen standen wohl die Fernkräfte bei *Laplace* und dessen Zeitgenossen. Faradays naiv-geniale Auffassungen und Maxwells mathematische Formulierung derselben haben die Berührungskräfte wieder in den Vordergrund gedrängt." [Mach, Mechanik, p. 185]

[44] The rise of relativistic physics began in the second half of 19th century. In 1903, Mach summarized the state of art being then in force. For his contemporaries, Mach said, it was strange that the notion "absolute motion" is a meaningless notion without content which cannot be used in science: "Die Ansicht, daß die 'absolute Bewegung' ein sinnloser, inhaltsleerer und wissenschaftlich nicht verwendbarer Begriff sei, die vor dreißig Jahren fast allgemein Befremden erregte, wird heute von vielen und namhaften Forschern vertreten. Ich möchte als entschiedene 'Relativisten' nur anführen: Stallo, J. Thomson, Ludwig Lange, Love, Kleinpeter, J. C. Mac Gregor, Mansion, Petzoldt, Pearson." [Mach, *Mechanik*, p. 233] Obviously, Mach neither relied on Leibniz nor on Euler. J. B. Stallo (1823–1900) was an American Judge and Ambassador to Italy. In 1880, he published a book on *The concepts and theories of modern physics* which was translated into German and edited by H. Kleinpeter with a foreword by Mach in 1901. On Stallo's life in America is reported in the webpage on *German element in Cincinnati*: The most remarkable man among the German lawyers of Ohio, "a man of whom all the Germans in the United States should be especially proud is Johann Bernhard Stallo. He came from a race of school-masters, and was born in 1823, in the Grand Dukedom of Oldenburg, and came to Cincinnati in 1839, where he was first a teacher in a private school when he compiled a German A, B, C, spelling-book, a great want, the superior merits of which led the directors of the newly founded Catholic St. Xavier's College to appointed him a teacher in that institution. The study of higher mathematics led him to German philosophy, and in 1848 appeared his *General Principles of the Philosophy of Nature* and in 1882 his *Concepts and Theories of Modern Physics*." [http://www.rootsweb.com/∼ohhamilt/ howe/847.html]

6.4.1 Postulated Simultaneity: Newton

As in Newton's theory absolute space and time were assumed to be independent of motion, now, emission and velocity of light were assumed and, moreover, proved to be independent of the dynamical state of the emitting light source. After the introduction of the gravitation constant, the invariance of numerical value c of light velocity was an additional hint of the existence of a new type of physical quantities, nowadays called *fundamental constants*. A preliminary version of relations of such type in the theory of motion had been introduced by Euler who assumed that the "increment of velocity is independent of the velocity" [Euler E015/016], [Euler E289] (compare Chap. 4). This independence can be regarded as a universal property of all bodies. Then, assuming Euler's model of resting and moving observers, who have proved the *invariance* of the equation of motion, now the observers have to make a decision between two methods to ensure the common experimental basis, either to preserve the equation of motion or to assume the invariance of light velocity. Einstein demonstrated that the principle of relative motion remains to be valid also in the model where the light velocity as an upper limit for the velocities of moving bodies is introduced.[45]

The modification concerns Euler's assumption on the *independence of the increment of velocity on the velocity* of a moving body. The change of the state expressed in terms of dv is the same for rest and motion, i.e. it is independent of the velocity the body. For the same force K, the same mass m and the same time interval dt, the equation of motion is given by the relation $dv_{\mathrm{rest}} = dv_{\mathrm{motion}}$. Now, Euler's basic assumption is to be replaced with the relation $dv_{\mathrm{rest}} \neq dv_{\mathrm{motion}}$. Therefore, each of two observers who are moving relatively to each other uniformly with the velocity v_{rel} in the opposite direction along the same straight line could be sure that the other observer obtained the same result for the change of the state of the moving body. Considering the light velocity, both the observers can be sure that the other is using the same numerical value of the light velocity. Nothing has to be modified as long as the light velocity is not considered as an *upper limit* for the motion of bodies. Rejecting the upper limit we obtain the relations

$$v_0 + dv_{\mathrm{rest}} = dv_{\mathrm{rest}},\, v + dv_{\mathrm{motion}} = v + dv_{\mathrm{rest}},\, dv_{\mathrm{rest}} = \frac{K}{m}dt \tag{6.3}$$

with $v_0 = 0$. The only invariant parameters of the system are the velocity of light c and the relative motion v_{rel}. Simultaneity is ensured by reference to Newton's absolute time which had been implicitly assumed for the derivation of the equation of motion. However, the *postulated* simultaneity had to be confirmed experimentally by the submitted sequence of signals between observers. Following Euler, the states

[45] "Hier setzt die Relativitätstheorie ein. Durch eine Analyse der physikalischen Begriffe von Raum und Zeit zeigte sich, *daß in Wahrheit die Unvereinbarkeit des Relativitätsprinzips mit dem Ausbreitungsgesetz des Lichtes gar nicht vorhanden sei*, daß man vielmehr durch systematisches Festhalten an diesen beiden Gesetzen zu einer logisch einwandfreien Theorie gelange." [Einstein, Allgemeine Relat]

of the bodies cannot be distinguished by the different velocities of the bodies. By this supposition, any ordering between bodies is excluded as far as the changes of their states with respect to a force of the same magnitude are concerned. The efficiency of the force in generating an increment of velocity is always the same.

Euler postulated that the change of velocity is *independent* of velocity. Hence, the alternative non-Eulerian postulat is represented by the relation

$$d\left[vf(v)\right] = \frac{K}{m}dt \tag{6.4}$$

where the dependence of the change of state depends on the velocity and is represented by a dimensionless factor which depends on the velocity $f(v)$. The efficiency of the force may be vary as follows $0 \leq f(v) \leq 1$ in dependence on the velocity. Obviously, the Eulerian relation is obtained for $f(v) = 1$ in the case of rest $v = 0$ whereas in case of motion, i.e. $v \neq 0$, the function is also different from the previous value for the case of rest $f(v) \neq 1$, i.e. the Eulerian case $v = 0$ is assumed to be the upper limit of the efficiency. Following Euler's interpretation, the *minimal* forces [Euler E343, Lettre LXXVIII] generate a *maximal* effect.[46]

The shape of the function $f(v)$ is readily obtained from the solutions of a differential equation

$$d\left[vf(v)\right] = f(v)dv + v\frac{df}{dv}dv = f^{\alpha}(v)dv \tag{6.5}$$

which follows from Eq. (6.4). The general solution of the differential equation is

$$f(v) = f(0)\left(1 - v^{\alpha-1}\right)^{\frac{1}{-\alpha+1}} \qquad \alpha = 2,3,4,\ldots \tag{6.6}$$

with $f(0) = 1$ which can be now specified for different values of the parameter α [Suisky 2005b]. Then, an additional parameter v_0 had to be introduced to ensure that the function remains to be dimensionless.

$$\begin{aligned} f_{\alpha=2}(v) &= f_2\left(\frac{v}{v_0}\right) = \frac{1}{1-(v/v_0)} \\ f_{\alpha=3}(v) &= f_3\left(\frac{v}{v_0}\right) = \frac{1}{\sqrt{1-(v/v_0)^2}} \end{aligned} \tag{6.7}$$

The introduction of this parameter is a consequence of the modification of Euler's postulate about the independence of the *change of the state* from the state. The function $f_3(v/v_0)$ fulfils the requirement for an ordering relation if the magnitude of the velocity is confined to the interval $0 \leq v < v_0$ or $1 \geq f(v) > 0$. Hence, an upper limit of velocity is defined whose existence is manifested in an ordering of the changes which are caused by an external force.

[46] However, Euler concluded that the *minimal* forces generate a *minimal* change of the state [Euler E343, Lettre LXXVIII]. This result is only valid if the change of the state does not depend on the state.

Conversely, it can be demonstrated that the function $f_3(v/v_0)$ is obtained from a supposed order between *temporal* and *spatial* intervals which are measured by observers who are resting or moving uniformly relative to each other. Now, instead of the efficiency of force, the system of resting or moving bodies in the absence of forces is investigated. Presence or absence of order is to be defined with respect to the magnitude of temporal and spatial intervals which are measured by different observers. As before in case of the change of the state, the magnitude of the temporal and spatial intervals had to be related to state of rest and compared to their change in case of uniform motion. Forces or the change of the state are not taken into account. Hence, there is either a maximal or a minimal temporal interval related to the state of rest. The same relation is supposed for the spatial intervals. Then, it is possible to define ordering relations and invariant expressions for the transformation between the measurements of different observers. Galileo-invariant and Lorentz-invariant transformations are obtained in case of absence and presence of ordering relation, respectively. Furthermore, the supposition of ordering relations results in an upper limit for the relative velocity $0 \leq v < v_0$ which is in agreement with the result derived before [Suisky 2006].

The introduction of an upper limit of velocity is a natural consequence of a generalization of Euler's mechanics. The crucial point is the consistent definition of the velocity as a physical quantity. Already in 1736, Euler formulated the basic principle that the *change of velocity is independent of the velocity* [Euler E015/016, § 131]. However, although it had not been mentioned, Euler excluded an upper limit of velocity by this procedure and treated the velocity as a *mathematical* quantity. Mathematically, the velocity can be increased in infinity, i.e. without limit. This is just the definition of a mathematical quantity given by Euler.

> No matter what kind of quantity it may be, we should understand that every quantity, no matter how large, can always be made greater and greater, and thus increased without limit, that is, increased to infinity. [E212, § 72]

Already in classical mechanics, the mathematical and the physical interpretation of velocity can be demonstrated by the *absence* and *presence* of order. Leibniz insisted that there are not only spatial and temporal orders, but (i) space and time are orders of different kind and (ii) both the orders are correlated. However, it was not possible to demonstrate quantitatively and physically this correlation. Following Newton, the absence of correlation between the orders necessarily results in absolute space and absolute time.

6.4.2 Experimentally Confirmed Simultaneity: Einstein

In 1905, Einstein established a model of relative motion making of those two non-interaction bodies and those two communicating observers who fit exactly the requirements Euler's world model from 1750. Analyzing the communication between the observers BOB and BOB', Einstein extended Euler's model by the inclusion of light as the third item beside bodies and observers whose properties are decisive for the communication between observers. However, Euler has already made use of this third item for the determination of parallel directions [Euler E842, § 77].

Obviously, the parallel lines are modelled by light rays which propagate in space and the information for the observers is transmitted by light. Moreover, Euler was fully aware of the retardation in transmitting information due to the finite velocity of light [Euler E343, Lettre XX]. The velocity of light had been measured by Ole Roemer (1644–1710) in 1675. Euler emphasized that there is always a delay in time of transmission of information to any observer living at a distant place with respect to the light source. He discussed such a configuration of things related the simultaneous creation of the earth and the stars 6000 years ago by God and the knowledge we obtain from them by the light transmitted through the universe to us or to our ancestors. As a consequence, despite the huge velocity of light, it was impossible for Adam to perceive the Garden Eden and the light of all stars at the same time [Euler E343, Lettre XX]. Furthermore, Euler stressed the similarity between sound and light to explain the enormous value of the speed of light comparing elasticity and the densities of air and ether to question Newton's theory of emanation [Euler E344, Lettres XVI–XXI].[47] Hence, Euler made use of the properties of light to explain *retardation* effects. Euler concluded that the retardation has to be included in the definition of simultaneity. Although God simultaneously created the stars, Adam perceived their light not simultaneously.[48]

Einstein concentrated on the definition of *simultaneity* yEinstein, Allgemeine Relat). Analyzing the communication between the observers in detail, Einstein concluded that the definition of *simultaneity* is only possible if the observers are making use of both parameters, the relative velocity and the velocity of light. Then, they can connect the frames of reference and the measurements of BOB1 and BOB2 in their labs. Hence, the fourth item added by Einstein is the analysis of the tools the labs made up, i.e. the units of time and the units of length. Obviously, this procedure is in contradiction to Newton's assumption that the invariant connecting parameter is the absolute time. Following Newton, coordinates and times, measured by different observers BOB and BOB', are related to each other by the Galileo transformation. The invariant parameters are t and v_{rel}.

$$x' = x - v_{\mathrm{rel}} \cdot t, \qquad t' = t. \tag{6.8}$$

[47] "III. Of Sound, and its Velocity, (...) XVII. Of Light, and the systems of Descartes and Newton, XVIII. Difficulties attending the System of Emanation, XIX. A different System respecting the Nature of Rays and of Light proposed. XX. Of the propagation of Light, XXI. Digression on the Distances of the Heavenly Bodies, and on the Nature of the Sun, and his Rays." [Euler E344, (New York 1837)]

[48] The light of the stars moves with as great velocity as the light of the sun. "Et si au commencement du monde les étoiles avoient été créées à peu près en même tems qu'Adam, il n'auroit pu les voir qu'au bout de 6 ans, et même celles qui sont les plus proches; car pour les plus éloignées, il lui auroit fallu attendre d'autant plus de tems, avant que de les découvrir." [Euler E344, Lettre XX] Moreover, Euler claimed that the same retardation may also experienced by us.

In accordance to Euler's assumptions [Euler E149], [Euler E842], motion is *relative* whereas time is *absolute*, but these different assumptions on motion and time cannot be preserved if motion, time and space are to be treated on an equal footing. Moreover, already Euler's postulate on space and time "being ideas of the same sort" [Euler E149, § 18][49] cannot be maintained since the transformation rules are different in Eq. (6.8). Following Einstein, the problem is to be solved by a rightfully defined simultaneity.[50] The simultaneity has to be probed by the exchange of information between the observers which is only limited by the finite velocity of light. Now, supposing that light velocity is not only the *upper limit* for the transmission of signals by light denoted by c, but also the upper limit for the *motion of bodies*, it follows $v_{\mathrm{rel}} < c$. This relation is in contradiction to Newton's absolute time and Leibniz's theory of the unlimited increase of velocity (compare Chap. 2). As a consequence, the light velocity c and the relative velocity v_{rel} are the only invariant parameters of the system since it is excluded that a body can move faster than light. Assuming two observers who are moving uniformly relative to each other, the ratio between these parameters is represented by a dimensionless real number z

$$v_{\mathrm{rel}} = z \cdot c, \tag{6.9}$$

whose value depends only on the choice of v_{rel}. Obviously, for a given constant relative velocity the number z is also constant. Both the parameters are *finite* quantities. Then, all other quantities should depend on z, or, generally, on a quantity which is a function of z. From Eq. (6.8) we obtain

$$x' = x - v_{rel} \cdot t = f(z)(x - z \cdot c \cdot t) \tag{6.10}$$

$$t' = f(z) \cdot t \tag{6.11}$$

with $f(z) = 1$ for $z = 0$. Therefore, the case of *relative rest* is described by the choice of the parameter $z = 0$. Generally, the parameter is varying in the interval $0 \leq z \leq 1$. For $z > 1$ i.e. $v_{\mathrm{rel}} > c$ the communication between the observers is impossible. The equations (6.10) and (6.11) are related to each other by the function $f(z)$. For $z \neq 0$, i.e. $f(z) \neq 1$, the second equation of (6.10) is not valid. Therefore, it has to be completed by additional terms having the same analytical form like those in Eq. (6.10). This conclusion results from the investigation of the invariance of the equation of motion for uniformly moving bodies. The requirement of invariance is transferred to the relation between time and space and the dependence on the parameters v_{rel} and c. The coordinate related part of Eq. (6.8) had been already obtained

$$x' = f(z)(x - z \cdot c \cdot t) \tag{6.12}$$

[49] "18. The ideas of space and of time have almost always been of the same sort, so that those who have denied the reality of the one have also denied of the other, and conversely." [Euler E149, § 18 (Uchii)]

[50] Then, Euler's postulate on the *increment* of velocity is to be replaced with the *experimentally* confirmed assumption that the velocity of light is independent of the velocity of light source. "*De Sitter* (konnte) zeigen, dass die Fortpflanzungsgeschwindigkeit des Lichtes von der Bewegungsgeschwindigkeit des das Licht emittierenden Körper unabhängig ist." [Einstein, Allgemeine Relat]

since it depends on v_{rel} whereas the time related part was independent of v_{rel} and, consequently of z, as yet. Now, making use of the advantages of Euler's *analytical* representation of mechanical quantities, it is due to the equivalence of the "quantities of the same sort" [Euler E149, § 18] that the *time* related part must take the same *analytical form* as the *coordinate* (position) related part has. The coordinate related part (6.10) consists of two terms, therefore, the time related part is to be also made up of two terms. Then, it follows

$$t' = f(z) \cdot \left(t - \frac{z}{c}x\right). \tag{6.13}$$

As far as mathematics is concerned, Euler claimed that *infinitesimal* time intervals and *infinitesimal* space intervals should be *mathematically* characterized by the same properties.[51] Now, this principle is also applied to *finite* space and time intervals and, therefore, the same function $f(z)$ is to be introduced in both equations (6.12) and (6.13).

Assuming an orthogonal transformation, the function $f(z)$ follows from equations (6.12) and (6.13). We obtain

$$f(z) = \frac{1}{\sqrt{1-z^2}} = \frac{1}{\sqrt{1-(v_{rel}^2/c^2)}} \tag{6.14}$$

which is just the factor known from the Lorentz transformation. A communication between the observers is only possible for $z < 1$. The equations (6.12) and (6.13) including their inverse transformation are only valid if both the observers are using the *same numerical* value of the velocity c. All other quantities cannot be considered as invariant system parameters. Therefore, neither time intervals (duration) nor space intervals (distances) are invariant for those observers who are moving uniformly relatively to each other with the velocity $0 < v_{\text{rel}} < c$.

6.4.3 Minkowski's World of Events

Later in the 20th century, the idea of instants and positions had been united and merged. The outcome is the idea of "events" the world consist of [Minkowski, Absolute world]. This world may be understood as a continuum of events which is free of other things, in first respect free of bodies. The only basic items are the relation between different events. Hence, an extended region of the world being occupied by a body has to be represented either as an infinity of events whose number is uncountable or by a body of infinitesimal extension. The body is to be described by a mass density of infinite magnitude. Minkowski introduced a relation between infinitesimal spatial and temporal intervals,

[51] "18. Cum loci ideam definiverim, prout eam quidem sensuum iudicium suppeditat, idea loci nunc quoque temporis, quae in notione quietis ac motus implicatur, occurrit. (...) Tempus igitur perinde nobis liceat in calculum introducere, ac lineas aliasque magnitudines geometricas." [Euler E289, § 18]

$$ds^2 = c^2 dt^2 - dx^2 \tag{6.15}$$

where the quantities ds, dx and dt are not treated as *derivatives*, but as *differentials*, i.e. by the same arithmetical methods which have been introduced by Newton, Leibniz and Euler [Euler E212].

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. [Minkowski, Space and Time, p. 75]

A world consisting only of bodies resting relatively to each other is excluded form the possible sets of worlds. Although Minkowski based his considerations on space and time, this world can be considered as *possible* world in the frame of the Leibnizian model of an "infinity of possible worlds", but is beyond Leibniz's construction of relational concepts of time and space.

> Since the [relativity] postulate comes to mean that only the four-dimensional world in space and time is given by phenomena, but that the projection in space and in time may still be undertaken with a certain degree of freedom, I prefer to call it the *postulate of the absolute world*.
>
> Neither Einstein nor Lorentz made any attack on the concept of space. We should then have in the world no longer *space*, but an infinite number of spaces, analogously as there are in three-dimensional space an infinite number of planes. Three-dimensional geometry becomes a chapter in four-dimensional physics. Now you know why I said at the outset that space and time are to fade away into shadows, and only a world in itself will subsist. [Minkowski, Space and Time, pp. 79–83]

Minkowski's approach had been developed and generalized by Weyl [Weyl, Raum und Zeit]. In 1924, Schrödinger made use of Weyl's theory to construct a function being a mathematical representations of a system made up of an electromagnetic field and a travelling electron which periodically, i.e. after a finite time, returns to its initial position [Schrödinger, Quantenbahnen]. The function Schrödinger obtained may be considered as a forerunner of the "wave function" which had been later obtained as solutions of the Schrödinger equation in 1926 [Schrödinger, Second Announcement].

Chapter 7
Euler's Wirksamkeit, Helmholtz's Treatment of Energy Law and Beyond

In the 17th century, Newton and Leibniz explained the key features of mechanics by reference to the distinction between static or the science of the Ancients and mechanics or the science of motion. The first decisive step had been done by Galileo who experimentally investigated the motion of falling bodies. Newton introduced the general relation between the "change in motion" and the "impressed moving force" whereas Leibniz, following Descartes who postulated the conservation of motion, emphasized the conservation of "living forces" (compare Chaps. 1, 2 and 4). In the end of the 19th century and the beginning of the 20th century there was not only a discovery of Leibniz's papers on logic by Couturat and Russell, but also an acknowledgement of Leibniz's contributions to mechanics. Helmholtz (1821–1894) and Planck (1858–1947) referred to Newton and Leibniz[1] in the foundation of mechanics and credited the invention of new mechanical principles as well the creation of the calculus.[2]

Helmholtz's *Vorlesungen über die Dynamik discreter Massenpunkte* had been published posthumously in 1898 based on the lectures given from December 2, 1893 to March 4, 1894. Obviously, Euler's notion of mass point had not only survived, but the theory was flourishing. Moreover, Helmholtz indirectly confirmed that Euler's and Weierstraß's foundation of the calculus turned out to be compatible to each

[1] Newton's program to determine the forces from the phenomena (sichtbare Wirkungen) and Leibniz's principle of sufficient reason interpreted in terms of causality: "Der theoretische Theil derselben sucht dagegen, die unbekannten Ursachen der Vorgänge aus ihren sichtbaren Wirkungen zu finden; er sucht dieselben zu begreifen nach dem Gesetze der Causalität. Wir werden genöthigt und berechtigt zu diesem Geschäfte durch den Grundsatz, das jede Veränderung in der Natur eine zureichende Ursache haben müsse. (...) und so fort, bis wir zu letzten Ursachen gekommen sind, welche nach einem unveränderlichen Gesetz wirken, welche folglich zu jeder Zeit unter denselben äusseren Verhältnissen dieselbe Wirkung hervorbringen. Das endlich Ziel der theoretischen Naturwissenschaften ist also, die letzten unveränderlichen Ursachen der Vorgänge in der Natur aufzufinden." [Helmholtz, Kraft]

[2] "(...) the elementary quantum of action plays a fundamental part in atomic physics and that its introduction opened up a new era in natural science, for it heralded the advent of something entirely unprecedented and was destined to remodel basically the physical outlook and thinking of man which, ever since Leibniz and Newton laid the ground work for infinitesimal calculus, were founded on the assumption that all causal interactions are continuous." [Condon]

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other when mechanically interpreted. Using Weierstraß's foundation, nothing had to be modified in the basic laws of mechanics.[3]

In Chaps. 4 to 6 it had been demonstrated that Euler invented a new type of representation of the laws and theorems of mechanics which had been developed by Galileo, Kepler, Descartes and Newton. Following the proposal of Johann Bernoulli, the new representation had later been called "dynamics" to accentuate the contrast to the science of the Ancients known by the name "static" [Euler, Opera Omnia, VI, 1, Preface IX and X (Stäckel)]. This name had been preserved in the further development of mechanics until now. Furthermore, Helmholtz substituted "static" with "kinematics" which comprises the mathematical representation of motion.[4] Following Helmhotlz, kinematics is composed of geometry and time where, following Newton, the latter is considered as a "measurable variable mathematical quantity".[5] Hence, Mach's criticism of Newton's concepts of absolute time is indirectly taken into account by accentuating that time is a "measurable" quantity.

7.1 Helmholtz' Treatment of Newton's Laws

Following the Newtonian scheme represented by the 1st, 2nd and 3rd Laws [Newton, Principia], Helmholtz subdivided mechanics into three parts, Part 1 Kinematics of a material point, Part 2 Dynamics of a material point and Part 3 Dynamics of a system of mass points [Helmholtz, Vorlesungen]. Obviously, Newton's "body" had been replaced with Euler's "mass point", but Helmholtz developed the theory rather in accordance with Newton's than with Euler's scheme based on the concept of mass points and the generation of forces. Following Euler, it is misleading to consider the dynamics of only one body ignoring the existence of other bodies. The introductory Chapter of the treatise *Theoria* [Euler E289] is entitled *Continens illustrationes et additiones necessarias de motu punctorum*, therefore, it is *not* devoted to the dynamics of a *single* mass point, but it is related to the motion of system of mass points, at least there are two mass point. Furthermore, Helmholtz used three names

[3] "Übrigens kommen diese naiven Betrachtungsweisen auch heute noch unwillkürlich zur Geltung, wenn man in der *mathematischen Physik*, der *Mechanik*, der *Differentialgeometrie* irgendeinen mathematischen Ansatz zustande bringen will. Sie alle wissen, daß sie überall da äußerst zweckmäßig ist. Freilich spottet der reine Mathematiker gern über eine solche Darstellung; als ich studierte, sagte man, daß für den Physiker das Differential ein Stück Messing sei, mit dem erwie mit seinen Apparaten umgehe." [Klein, Elementarmathematik, p. 227]

[4] "(...) müssen wir eine Erörterung voranschicken, in welcher wir diejenigen Begriffe aufstellen, die zu einer mathematischen Darstellung der Bewegungserscheinungen geeignet sind. Die Lehre von der Aufstellung und dem Zusammenhange dieser Begriffe nennt man *Kinematik*." [Helmholtz, Vorlesungen, § 1]

[5] "Bewegung nennen wir die Ortsveränderung einer Masse in der Zeit. Wir werden daher in der Kinematik außer den rein geometrischen Begriffen, welche zur Angabe der Orte, Ortsveränderungen und Wege erforderlich sind, auch die fortschreitende Zeit als eine meßbar veränderliche mathematische Größe mit in die Betrachtung ziehen müssen." [Helmholtz, Vorlesungen, § 1]

for the objects which are considered in mechanics, first, in the title "discrete mass points" (discrete Massenpunkte), second, in the headings of the Chapters "material points" (materielle Punkte) and third in the text "mass point" (Massenpunkt) which is synonymous with "material point". Motion is defined as the change of place of a mass in time.

> "Die Dynamik umfasst die Lehre von denjenigen Naturerscheinungen, welche zurückzuführen sind lediglich auf die Bewegung ponderabler Massen. ... Bewegung nennen wir die Ortsveränderung einer Masse in der Zeit. Die Massen, deren Bewegung wir verfolgen können, haben stets eine räumliche Ausdehnung und eine bestimmte geometrische Gestalt, (...)." [Helmholtz, Vorlesungen, § 1]

Obviously, Helmholtz replaced the traditional notion of *body* used in mechanics by the notion of *mass*. In Newton's and Euler's mechanics the laws are formulated for the rest and motion of bodies. Instead of rest and motion traditionally assigned to a body, Helmholtz did introduce first a *mass* which is changing its position and second the spatial extension and shape which related to the mass. "Dynamics comprises the theory of phenomena which are only related to the motion of weighty masses. Motion is the change of the place of a mass in time. The masses have always a spatial extension and a figure (shape)" [Helmholtz, Vorlesungen, § 1]. The extension of the body is only mentioned as an additional property. The mass point is defined by a mass which spatial extension and shape can be neglected, which position can be, consequently, described by a point. However, the corporeal masses are needed for the introduction of mass points. The notion of mass points simplifies the calculations.

> (...) wir wollen uns jetzt zur Vereinfachung die körperlichen Massen (...) als materielle Punkte vorstellen. [Helmholtz, Vorlesungen, § 2]

The mass *point* is defined *geometrically* by comparison of two bodies of different extension or by the relation of the extension of the body to the distances between the body and other bodies in the system. Instead of other bodies as the origin of forces, Helmholtz introduced given external forces and distinguished between these given external forces and the internal forces between the bodies (mass points) of a system [Helmholtz, Vorlesungen, § 39]. Obviously, instead of the Eulerian *unified* theory of forces, Helmholtz introduced *forces of different type* and *origin* as Leibniz did, but different from the Leibnizian scheme. There are forces whose origins are indeterminate, i.e. the origin is *outside* the system, and other forces whose origin is related to the mass points (bodies) of the system. Here, the Newtonian type attraction and repulsion is introduced as an indeterminate interaction between bodies. It is considered together with Eulerian type forces which are related to the interaction of bodies belonging to the system. The forces are subdivided into *internal* and *external* forces with respect to the system.[6]

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[6] Helmholtz did not make use of Euler's relational definition of forces, but referred to causality. "Diese objective Gesetzmäßigkeit nennen wir Kraft, und die Erforschung der Gesetze der tathsächlichen Bewegungen ist identisch mit der Beantwortung der durch unser Causalitätsbedürfnis geforderten Frage nach den Ursachen oder den Kräften. Dieser Auffassung

7.2 The Interpretation of the Calculus: Kinematics and Dynamics

Helmholtz's *Vorlesungen über die Dynamik discreter Massenpunkte* are in goal and spirit a continuation of Euler's mechanics of points of infinitesimal magnitude whereby, in contrast to Euler's approach, the mathematical part is related to the foundation of the calculus in terms of limits invented by Weierstraß. In this time, another essential novelty Euler had essentially contributed to, the notion of function, is generally accepted and thought to be the indispensable basis of mathematics. Nevertheless, in the end of the 19th century, Weierstraß's mathematical innovation was not generally accepted by physicists although the progress in the foundation of the calculus had been emphasized by mathematicians. Moreover, the merits of Euler as the inventor of this development in the 18th century had been highly acknowledged.[7]

In the *Vorlesungen*, Helmholtz interpreted the calculus in terms of Weiserstraß's foundation by limits. Having introduced the notion of position, velocity and acceleration by notions known from the 17th and 18th century mechanics as "Aenderungsgeschwindigkeiten der Geschwindigkeitscomponenten", in former times also called differentio-differentials,[8] Helmholtz stated:

> (...) wenn wir die Operation
>
> $$\frac{d\left(\frac{dx}{dt}\right)}{dt} \text{ oder in kürzerer Form } \frac{d^2x}{dt^2}$$
>
> anwenden, deren Resultat man den zweiten Differentialquotienten von x nach t nennt. (...) Die Beschleunigungen der Coordinaten sind durch deren zweite Differentialquotienten nach der Zeit gegeben. [Helmholtz, Vorlesungen, § 6]

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entstammt der Name Dynamik, d.i. Lehre von den Bewegungskräften. (...) Daß eine Masse, welche eine Zeit lang in Ruhe existiert hat und dann plötzlich anfängt sich zu bewegen, durch irgendeine vorher fehlende Ursache dazu angetrieben werden muß, das liegt schon in der Allgemeingültigkeit des Causalitätsgesetzes, welches wir als Anschauungsform mitbringen. (...) jene Alten hielten im Allgemeinen an der von Aristoteles geäußerten Ansicht fest, dass jeder von der Ruhe verschiedene Bewegungszustand zu seiner Erhaltung einer dauernd wirkenden Ursache (Kraft) bedürfe." [Helmholtz, Vorlesungen, Zweiter Theil, § 8]

[7] "Vier Stufen werden wir unterscheiden, die durch die Namen Euler, Lagrange, Cauchy und Weierstraß charakterisiert sind. Sie bezeichnen die Hauptmomente in einer Bewegung, die eine immer weiter gehende logische Verschärfung bedeutet und von Herrn F. Klein (F. Klein, *Über die Arithmetisierung der Mathematik*, Göttinger Nachrichten, 1895) eine Arithmetisierung der Mathematik genannt worden ist. (...) Euler schafft den Boden für eine Arithmetisierung. Wenn wir, dem Herkommen folgend, die Periode Eulers als die naive bezeichnen, so sind wir uns der relativen Bedeutung diese Worte wohl bewußt." [Bohlmann] Obviously, Bohlmann claimed that this period should not be underestimated in its importance for the development in the 19th century.

[8] The Newtonian origin of the notion is obvious. Newton called these quantities *fluxions* and *fluxion of fluxions* (compare Chaps. 2 and 3) whereas, following Leibniz, the names of *differentials* and *differentio-differentials* had been used: "Begrifflich wären dieselben zu bezeichnen als die Aenderungsgeschwindigkeiten der Geschwindigkeitscomponenten, welch' letztere ursprünglich eingeführt waren als die Aenderungsgeschwindigkeiten der Coordinaten." [Helmholtz, Vorlesungen, § 6]

Obviously, following Weierstraß, Helmholtz exemplified the relation *mathematically*. The physical interpretation is based on a certain formal *mathematical* relation.[9] Following Helmholtz, the increase of velocity is *not disappearing* whereas the time particle *disappears*. Therefore, taking Helmholtz's statement literally, the "increase of velocity is appearing" whereas the "time particle is disappearing" [Helmholtz, Vorlesungen, § 12], but only in the limit. Helmholtz's explanation is as impressive as instructive since it demonstrated the difficulty to interpret *Weierstraß' algorithm* within a *mechanical* frame.[10] Mathematically, the second differential quotient is the result of a division of the numerator divided by the denominator. Helmholtz related to mathematical formalism to measurement, but runs into trouble to define the "smallness" operationally. Mathematically, there is no problem to assume a "quantity less than a given quantity" since neither the given nor the asked quantities are compared to experiment. Therefore, Helmholtz simultaneously made use of the Eulerian and the Weierstraßian method and terminology to define a "quantity being of *sufficient* smallness".

> Die Differencirbarkeit fordert mehr als die Stetigkeit; sie verlangt, daß die Veränderung, welche eine Function bei einen Zuwachs der Variablen erfährt, zu diesem Zuwachs in einem Verhältnis stehen muß, welches bei hinreichender Kleinheit dieses Zuwachses von dessen Größe unabhängig wird, also einen festen Grenzwert besitzt. [Helmholtz, Vorlesungen, § 4]

Following Euler, the indicated quantity is not only "sufficient small", i.e. "less than any assignable quantity", but also of *constant* magnitude [Euler E212] (compare Chaps. 4 and 5) whereas, following Weierstraß the magnitude in neither constant nor determinate, but can be chosen arbitrarily. As a consequence, Euler's procedure fits perfectly to the *constancy of units* underlying every measurement (compare Chap. 4). The differentials or increments of infinitesimal magnitude play the role of *ideal constant units*, forming arithmetical progressions of the independent variables

[9] Formal means that the definition of Weierstraß is not related to a mechanical model, but is completely independent of any mechanical foundation.

[10] The procedure had been already analyzed by Berkeley in 1734 before the rigorous foundation of the calculus. However, Helmholtz's interpretation of the Weierstraßian limit demonstrates that the mechanical model of the relation between (i) the *non-vanishing increment* of the *velocity* and (ii) the vanishing time particle is still unresolved as it was in 1734. Berkeley stated: "XIV. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step. From that Supposition it is that I get at the Increment of x^n, that I am able to compare it with the Increment of x, and that I find the Proportion between the two Increments. I now beg leave to make a new Supposition contrary to the first, i.e. I will suppose that there is no Increment of x, or that o is nothing; which second Supposition destroys my first, and is inconsistent with it, and therefore with every thing that suppose it. I do nevertheless beg leave to retain nx^{n-1}, which is an Expression obtained in virtue of my first Supposition, which necessarily presuppose such Supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity." [Berkeley, Analyst, XIV] However, Helmholtz's statement may be appropriate to confirm an early model established by Leibniz in 1672–75. "(…) a continuous line is composed not of points but of infinitely many infinitesimal lines, each of which is divisible and proportional to a generating motion at an instant (conatus)." [Arthur, Fictions]

(compare Chap. 5), being used for the quantitative interpretation of thought experiments using models such as the motion of mass points. As it had been discussed before, these progression of infinitesimal increments are readily obtained from experimentally confirmed arithmetical progressions which had been obtained from finite increments.

From the mechanical requirements it follows that an *increment* (decrement) of a function $f(x)$ of an independent variable x should be related to an *increment* (decrement) of the variable, i.e. $x \pm \Delta x$. There is no change of the function if the variable preserves one and the same value. Following Helmholtz, the coordinates or the path are functions of time $x = x(t)$. However, the inverse function where time is a function of coordinates, i.e. $t = t(x)$, is not considered its differential quotient has not been given by $\frac{dt}{dx}$. Thus, in contrast to Euler's treatment in the *Mechanica* [Euler E015/016, §§ 33–53], the inverse function is out the scope of mechanics. Euler assumed that there is a general relation between the differentials ds and dt since, for finite increments Δs and Δt, it holds $\Delta s = v \cdot \Delta t$. Nevertheless, Euler's model for the relation between differentials had been recovered by mathematicians. Minkowski assumed that space is related to time and time is related to space, $ds^2 = c^2 dt^2 - dx^2$ [Minkowski, Space and Time]. Furthermore, Helmholtz never completely abandoned the "old concept", but formulated essential relations in terms of the old-fashioned differentials. Discussing the essential question, whether the increment of velocity is independent of the velocity, Helmholtz stated:

> Eine zweite Frage, welche von *Newton* in der Abfassung seiner Axiome unberührt gelassen, dadurch freilich stillschweigend entschieden worden ist, betrifft den Einfluß der vorhandenen Geschwindigkeit. Die Größe der Kraft wurde gemessen allein durch die Masse und ihre Beschleunigung, und wenn auch die Beschleunigung definirt wurde durch den Grenzwerth des Zuwachses an Geschwindigkeit dividirt durch das verschwindende Zeittheilchen, in welchem derselbe zu Stande kommt, so ist doch dieser Begriff nicht abhängig von der Größe der bereits bestehenden Geschwindigkeit. Das *Newton*'sche Kraftmaaß verneint also den Einfluß der Geschwindigkeit. [Helmholtz, Vorlesungen, Zweiter Theil, § 12][11]

The first question was whether the influence of a "Bewegungskaft" (motion force) is not modified due the presence of another "Bewegungskraft". Here, the argumentation is taken from static and transferred to dynamics. The independence of the increment of the velocity on the velocity had been already assumed by Euler [Euler E015/016, § 131]. However, Helmholtz restricted the theorem on the discussion of the relation between velocity and acceleration whereas Euler demonstrated the mechanical background represented by the equivalence in the changes of the states of rest and motion.

Hence, following Helmholtz, the acceleration is obtained from "the limit of the increment of velocity *divided* by the vanishing (evanescing) particle of time in whose duration it is produced", i.e. assuming the arithmetical operation division,

[11] This result follows directly from the invariance of the equation of motion for Galileo transformations obtained already by Euler [Euler E842, Chap. IX], i.e. from the formal structure of the equation of motion, which had been also later discussed by Einstein. Euler studied the change of the state of a resting body due to an external cause (perturbation). Then, he transferred the relation for the change of the state of a *resting* body due the perturbation to the change of the state of a *uniformly moving* body which is caused by the same perturbation, i.e. by a force of the same direction and magnitude.

Helmholtz made use of the differentials as Leibniz and Euler did although he claimed that there is a limit. However, as it had been discussed in Chaps. 3 and 5, the inventors of the concept of limit intended to exclude the use of differentials. The idea of an "evanescent particle of time" ("verschwindendes Zeittheilchen") may be interpreted in terms of Leibniz's "evanescent quantities". The word "Zeittheilchen" for the quantity *dt* had been introduced by Wolfers for the translation of the word "tempusculum" [Euler E015/016 (Wolfers)]. The corresponding "smallest elements of the path" *ds*, ("die kleinsten Elemente des Weges") had been called "spatiolum" by Euler [Euler E015/016 (Wolfers) § 13]. These are the basic elements for the representation of motion whose geometric ratio $v = ds/dt$ is called velocity.

7.3 Helmholtz' Treatment of Leibniz's "Living Forces"

Helmholtz's derivation of the energy conservation law is appropriate to demonstrate how the correlation between mathematics and physics can be used for the invention of reliable algorithms for the representation of experimental findings. Helmholtz discussed the relation between the geometry of a system and the corresponding differential equations. Geometry is represented either be the configurations of the system or the paths of the bodies being parts of the mechanical system. As it will be discussed in the next Chapter, the *configurations* and the *configurational space* play an important role in Schrödinger's foundation of quantum mechanics [Schrödinger, Second Announcement]. Helmholtz considered periodic motion, i.e. the return of a system into its initial configurations, which had been already discussed by Leibniz. Furthermore, Helmholtz demonstrated a new technique for the integration of Newtonian equation of motion and, moreover, a ingenious method for the definition of new mechanical quantities [Helmholtz, Vorlesungen, §§ 22, 48].[12]

> Leibniz stellte nun folgenden Satz auf (...), daß nämlich die Summe der lebendigen Kräfte in einem Massensystem allemal dieselbe wird, wenn die sämmtlichen Theile des Systems im Laufe ihrer Bewegung in die gleiche Lage zu einander zurückkehren. Diese Gesetzmäßigkeit bezeichnete er als conservatio virirum vivarum, er ging also nicht auf den Begriff der potentiellen Energie ein. (...) Wenn nämlich die Summe der lebendigen Kräfte für jede während der Bewegung wiederkehrende Constellation der Massen auch wieder den gleichen Werth erlangt, so muß dieselbe, obwohl sie nur aus den Massen und den Geschwindigkeitsquadraten zusammengesetzt ist, doch eine reine Function der Coordinaten der Massenpunkte sein. [Helmholtz, Vorlesungen, § 48][13]

[12] Helmholtz considered motion along a straight line [Helmholtz, Vorlesungen, § 22]. Helmholtz claimed that the nature of motion is clarified by the integration of the equation of motion. "Die Integration dieser Differentialgleichung muß uns nun über die Natur der Bewegung belehren, welche der Massenpunkt unter der Wirkung der elastischen Kraft $X = -a\,a\,x$ ausführt." [Helmholtz, Vorlesungen, § 22] Helmholtz made us of an "integrating factor", in the present example the integrating factor is the velocity dx/dt, instead of the direct integration of the differential equation. Also in § 22, Helmholtz presented an interpretation of the "differential quotient" in terms of limit (compare Sect. 7.2).

[13] Already in 1847, Helmholtz referred to Leibniz. "I. The principle of the conservation of living force. II. The principle of the conservation of force." [Helmholtz, Kraft]

Introducing the theorem that "the sum of living forces should be expressed in terms of a pure functions of coordinates" Helmholtz did an essential step beyond Leibniz. Nevertheless, Helmholtz's theorem is in complete agreement with Leibniz's supposition that mechanics is obtained by completion of geometry by principles specifying the action and the resistance of bodies. Hence, although mechanics cannot solely be founded on geometry [Leibniz, Specimen, I (11)], there should be a conformity between geometry and dynamics. The dynamics of the system, represented by living forces, cannot be decoupled from the occupation of places in certain configurations by the moving bodies. Then, the sum of the kinetic energies of the involved bodies turns out to be a pure function of the relative positions of the bodies within the system, i.e. it is a *pure function of coordinates* [Helmholtz, Vorlesungen, § 48]. Helmholtz claimed that the existence of such a function has to be required. However, the special shape of these functions does not follow from the requirement, but additional assumptions on the relation of that function to other mechanical quantities different from total energy and kinetic energy, especially to forces are needed.

Helmholtz considered a system of moving bodies whose interactions are described by purely coordinate dependent forces.

$$\Sigma M_a \frac{d^2 x_a}{dt^2}\frac{dx_a}{dt} = \Sigma K_a \frac{dx_a}{dt} \tag{7.1}$$

The l. h. s. of this equation can be written as the differential quotient of the sum of "living forces", nowadays called "kinetic energy", of all bodies of the system.

$$\frac{d}{dt}\Sigma\frac{1}{2}M_a\left(\frac{dx_a}{dt}\right)^2 = \frac{dL}{dt} = \Sigma K_a \frac{dx_a}{dt} \tag{7.2}$$

Then, making use of the representation of forces by the derivatives of the potential Φ (this quantity had been called "Wirksamkeit" or "efficiency" by Euler and labelled by the same letter Φ [Euler E197], compare Sect. 7.5), it follows:

$$\frac{d}{dt}\Sigma\frac{1}{2}M_a\left(\frac{dx_a}{dt}\right)^2 = \frac{dL}{dt} = \Sigma K_a \frac{dx_a}{dt} = \Sigma\left(-\frac{\partial\Phi}{\partial x_a}\right)\frac{dx_a}{dt} = -\frac{d\Phi}{dt} \tag{7.3}$$

Helmholtz assumed that the coordinate dependent function is represented by $-\Phi$ where the choice of the sign minus does not follow from the analytical treatment, but needs an additional justification obtained from the mechanical interpretation of the quantity Φ which will be given later.[14]

$$\frac{d}{dt}(-\Phi) = \Sigma\left(-\frac{\partial\Phi}{\partial x_a}\frac{dx_a}{dt}\right) \tag{7.4}$$

The result is presented in the compact form where differential of a sum made up of "living forces" and "efficiency" of living forces is equal to zero

[14] "(...) das Minuszeichen, welches zunächst willkürlich ist, aber für die analytischen Betrachtungen keineswegs anstößig erscheint, findet später seine Begründung in der physikalischen Bedeutung von Φ." [Helmholtz, Vorlesungen, § 48]

$$\frac{dL}{dt} = \Sigma X_a \frac{dx_a}{dt} \quad \text{and} \quad \frac{d}{dt}(L+\Phi) = 0 \tag{7.5}$$

and, consequently, the sum itself is constant by mathematical reasons.

$$L+\Phi = E = \text{const.} \tag{7.6}$$

The first part is determinate by the motion of bodies whereas the second part depends on the configurations of the system [Helmholtz, Vorlesungen, § 48]. The configurations of the system comprise *all places* the bodies attain by motion. Obviously, assuming $E = \text{const}$, there are configurations being not excluded by *externally* imposed constraints, but only by the *internal* constraint that one of the terms in the sum $E = L+\Phi$, either L or Φ, can only be increased if the other is diminished. Hence, for given E and L, the attainable configurations are to be compatible with the criterion established by the difference of total energy and living forces $\Phi = E - L$. Then at first, Helmholtz emphasized that Leibniz's requirement is fulfilled, i.e. the kinetic energy becomes a pure function of coordinates.

> In diesem Sinne sagt auch unser Resultat nichts mehr und nichts weniger aus als jener Leibniz'sche Satz; dieser verlangt ja, daß die lebendige Kraft durch eine reine Function der Coordinaten der Massenpunkte darstellbar sei, wir haben nun Φ als eine solche Function eingeführt und erhalten: $L = E - \Phi$, ein Ausdruck, welcher wegen der Konstanz von E diese Forderung erfüllt, ohne daß durch die besondere Form $E - \Phi$, d.h. durch die Heranziehung der potentiellen Energie eine darüber hinausgehende Erkenntniß erzielt wäre. [Helmholtz, Vorlesungen, § 48]

At second, Helmholtz stated that an additional insight is obtained by the mechanical interpretation of the function Φ, especially by the justification of the choice of its sign in Eq. (7.4). Until now, the choice of the sign is neither determinate by the analytical treatment of coordinate dependent functions nor by the analytical representation of "living forces".

7.4 The Extension of a System

Helmholtz demonstrated a new technique for the integration of Newtonian equation of motion and, moreover, an ingenious method for the definition of new mechanical quantities. The derivation of the energy conservation law is discussed for the Leibnizian model of a system which performs periodic motion reoccupies a given configuration after a certain finite time. In Sect. 7.3, it had been examined how Helmholtz introduced a purely coordinate dependent function to describe the conservation of living forces and finally obtained the general form of energy conservation law. Now, another essential consequence of Helmholtz's approach will be explored which concerns the *extension* of the system. Obviously, returning in its initial position, the system had not only conserved its energy, but also necessarily its extension is space. All positions are re-occupied by the same bodies, therefore, the extension in space related to the initial state is the same as the extension in space in the final state. Hence, Helmholtz formulated a conservation of the extension of the system whose extension can be only changed by an interaction with the environment being either accompanied with an accumulation or a loss of energy. Therefore, Helmholtz emphasized the relation between *energy* and *extension*.

There are two main configurations of a finite number of bodies. (I) There is no potential energy at all, the total energy consists of the kinetic energy of the bodies, in the absences of interactions (impacts) all bodies move along straight lines, the distances between them can be increased without limitation, the system has no invariant shape. (II) Following Leibniz, there are interactions between bodies which let the bodies return to their initial positions. The simplest examples for such systems are (i) the uniformly moving mass point and (ii) the linear harmonic oscillator, respectively. The uniformly moving mass point never returns to any of the positions it had traversed before whereas the moving mass of the oscillator returns to *each* of the positions it has traversed before. This feature may be considered as a *confinement* of motion to a certain region of the space. Consequently, the energy of the mass point is independent of coordinates, $E = (1/2)mv^2 = \text{const}$, whereas the energy of the oscillator, although it is related to coordinates is also constant and given by the sum of a coordinate and a momentum or velocity dependent function $E = (1/2)mv^2 + V(x) = \text{const}$.

The relation between extension and energy can be readily demonstrated by these models. Any *confinement* can only appear if the *coordinate* dependent function is related to energy and, vice versa, if the energy is related to a coordinate dependent function like $V(x)$ in the 1D case. Hence, Helmholtz's interpretation of Leibniz's model results not only in the definition of the total energy, but also in the introduction of confinement. In Leibniz's model of impact or in any other model of impact, confinement is only temporarily included since, after the impact, the bodies continue their motion according to case (I) moving uniformly in straight directions.

Helmholtz discussed two types of conditions, derived from two general principles. Both the principles operate independently of total and kinetic energies. The first type is based on the concept of time-independent paths which are distinguished from time-dependent trajectories. The bodies can be translated along arbitrarily chosen paths connecting different configurations and defining that function in the whole which has been previously related to kinetic energy. The second type of principles is based on Euler's method of maxima and minima and fixes special configurations of the system by the intrinsic features of the coordinate dependent function.

The relation to the coordinate dependent forces $K(x)$ in space is given by the integral relation

$$A = V(x_2) - V(x_1) = \int_1^2 dx \frac{\partial V(x)}{\partial x} = -\int_1^2 dx \cdot K(x). \tag{7.7}$$

Defining in that way the work A, the quantity $V(x)$ is considered as disposable work storage that is diminished or enhanced by giving or receiving energy to and from the environment, respectively. Since there is always a shift of the system or of a part of the system from the place x_1 to the place x_2, the *change of energy* is accompanied by a *change in the extension* of the system. The general condition valid for the arrangement of forces reads in the 3D case as follows:

$$\mathrm{rot}\vec{K} = \mathrm{rot}(\pm \mathrm{grad} V) = 0. \tag{7.8}$$

The criterion (7.8) is fulfilled independently of the choice of the sign for the gradient, i.e., the relative orientation of the force $\vec{K}$ and the *gradient* of the function $V(x,y,z)$ is not determinate by the mathematical relation which is fulfilled for different choices of the signs $\vec{K} = \pm \mathrm{grad} V(x,y,z)$ (compare [Helmholtz, Vorlesungen, § 6]).

Following Helmholtz, we have to chose the negative sign in order to explain the most essential mechanical features of the system, $\vec{K} = -\mathrm{grad} V(x,y,z)$, which follow from the relations between system and environment and make the difference to mathematics.

The following question are to be answered: Firstly, how is the total energy increased by an increase of A due to a supply of work from outside, and how is it diminished by a decrease of the same quantity due to giving work to the environment. Secondly, how is the work transformed into internal kinetic energy and vice versa, how is the internal work transformed into external kinetic energy. Then, considering both the processes, the total energy is represented by the sum $E = T(p) + V(x)$.

Moreover, the energy relation operates as a limiting function for the extension of the system by virtue of $E = T(p_{\mathrm{max}}) = V(x_{\mathrm{max}})$. Furthermore, assuming $p(t) = mv(t)$ and $v(t) = dx(t)/dt$ we obtain the Newtonian type of equation of motion $dp/dt = K$ which follows from $E = T(p) + V(x)$ and $\vec{K} = -\mathrm{grad} V$. Alternatively, preserving $p(t) = mv(t)$ and $v(t) = dx(t)/dt$, we obtain a non-Newtonian type of equation of motion $dp/dt = -K$ for $\vec{K} = +\mathrm{grad} V$ which follows from the relation by the choice of the other sign $E = T(p) - V(x)$ of the coordinate dependent function. Obviously, ignoring the relation between energy and forces and referring only to the coordinate and momentum dependent functions $T(p)$ and $V(x)$, respectively, we can only conclude that the total energy is made up by two additive components $T(p)$ and $V(x)$ whose physical meaning is not completely defined by the relation of their arguments $p(t)$ and $x(t)$. Hence, there is no criterion to distinguish between the representations $E = T(p) + V(x)$ and $E = T(p) - V(x)$. Both results are mathematically equivalent and are only based on the Leibniz-Helmholtz assumption on the representation of living forces by a purely coordinate dependent function.

The results can be summarized by a set of correlated relations between the quantities E, $T(p)$ and $V(x)$. The scheme represents the content of Helmholtz's treatment of the energy law and, at the same time, it demonstrates the step beyond.

Examining the functions for kinetic and potential energy independently of each other, it is immediately apparent that the functions $T(p)$ and $V(x)$ are defined in the *whole configuration* and the *whole momentum* space. The system can occupy any position in configuration space and any position in momentum space, i.e. $-\infty < x < +\infty$ and $-\infty < p < +\infty$, respectively. There are (i) no constraints defined by relations between coordinates which reduce the degrees of freedom for motion and (ii) there is no upper limit for the velocity (compare Chap. 6). The relations between

$T(p)$ and $V(x)$ are not fixed at yet whereas the relations between total and kinetic as well total and potential energy, i.e. E and $T(p)$ as well as E and $V(x)$, respectively, are defined by the requirement Helmholtz had introduced. Following Helmholtz and Leibniz, the kinetic energy is always greater than zero and either equal or less the total energy. The potential energy is always equal to or less than the total energy.

Then, the following relations between total and kinetic energy are obtained[15]:

$$\text{either } [E - T(p) \geq 0] \text{ or } [E - T(p) < 0], \tag{7.9}$$

$$[E - T(p) \geq 0] \quad \text{and} \quad [E - T(p) < 0], \tag{7.10}$$

$$\text{neither } [E - T(p) \geq 0] \quad \text{nor } [E - T(p) < 0]. \tag{7.11}$$

The corresponding relations between total and potential energies are of special interest since they describe the correlation between energy and extension or, the configurations of the system.

$$\text{either } [E - V(x) \geq 0] \text{ or } [E - V(x) < 0], \tag{7.12}$$

$$[E - V(x) \geq 0] \quad \text{and } [E - V(x) < 0], \tag{7.13}$$

$$\text{neither } [E - V(x) \geq 0] \quad \text{nor } [E - V(x) < 0]. \tag{7.14}$$

The classical harmonic oscillator is described by Eqs. (7.9) and (7.12). For a given finite total energy E_{total} the configuration space is divided into two regions, first those the system is subsequently occupying and those the system cannot occupy at all because Helmholtz's condition $E - V(x) \geq 0$ would be violated in this case. Therefore, the logical relation between the statements in Eqs. (7.9) and (7.12) is expressed in terms of an *exclusive* "either ... or ...". The system cannot attain all configurations, but only a certain subset formed by *possible* configurations which is the complement of the set of *impossible* configurations, labelled by C^{all}, $C_{\geq} = C_{\text{poss}}$ and $C_{<} = C_{\text{imposs}}$, respectively.

$$C^{\text{all}} = C_{\geq} \cup C_{<} \qquad\qquad C^{\text{all}} = C_{\text{poss}} \cup C_{\text{imposs}} \tag{7.15}$$

Taking into account all relations represented by of Eqs. (7.9) to (7.10), two kinds of systems can be defined resulting from the different assumptions:

(i) Classical Helmholtz case $\quad C^{\text{class}}_{\text{poss}} = C_{\geq}$

(ii) Non-classical Helmholtz case $\quad C^{\text{non-class}}_{\text{poss}} = C_{\geq} \cup C_{<}$

The classical systems are well-known, but they were mostly characterized by the *paths* rather than by *configurations*. Hence, a non-classical system is to be defined by the "absence" of paths, i.e. the impossibility to make use of the classical concept of localization [Heisenberg 1925] and the introduction of the whole configuration space [Schrödinger, Second Announcement] (compare Chap. 8). Schrödinger's introduction of the wave function being defined in the whole configurations space corresponds the non-classical Helmholtz case.

[15] Relations of the same type had been discussed by Euler [Euler E015/016], [Euler E289]. "3. Nullum enim existere potest corpus, quod non vel moveatur vel quiescat." [Euler E015/016, § 3]

In case (i) the system is localized in a certain subspace of the configuration space whereas in case (ii) this restriction had been removed, as long as theoretically, and the system is not localized in a certain subspace, but 'occupies' the whole space. The non-classical Helmholtz case cannot be modelled in the frame established as yet since the relations $E - V(x) \geq 0$ and $E - V(x) < 0$ cannot be simultaneously fulfilled without a reinterpretation of the kinetic energy. Following Helmholtz and Leibniz, the term $T(p)$ in the relation $T(p) = E - V(x)$ is to be reinterpreted and substituted by appropriated coordinate dependent functions $T(p) \rightarrow F(x)$ which are (a) defined in the whole configuration space as the function $V(x)$ and (b) ensure the validity of the relation $F(x) = E - V(x)$ for $E - V(x) \geq 0$ and $E - V(x) < 0$ by the corresponding relations $F(x) \geq 0$ and $F(x) < 0$, respectively. (c) There are no objections in advance that it is impossible to construct such functions which change their sign by passing from one region of the configuration space into another one. As it will be discussed in the Chap. 8, Schrödinger's amplitude function is appropriately defined for such purpose. As it had been mentioned above, Schrödinger introduced such a function which is defined in the whole configuration space and called it wave function.

7.5 Euler's *Wirksamkeit*

The notion of *Wirksamkeit* (efficiency, effort) had been developed by Euler in the complete conceptual frame of mechanics, i.e. for the case of motion in the 3D space by the generalization of the equation of motion along a straight line. Having derived the equations of motion of a body in the 3D space in terms of time dependence

$$mdu = Pdt \quad mdv = Qdt \quad mdw = Rdt \tag{7.16}$$

which had been completed by coordinate dependence

$$mudu = Pdx \quad mvdv = Qdy \quad mwdw = Rdz, \tag{7.17}$$

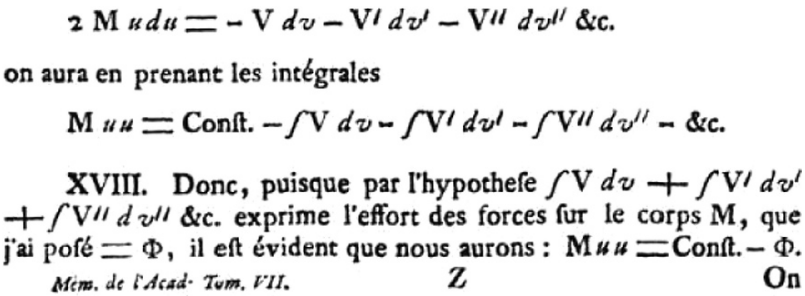
Euler analyzed the integral representation of these expressions (see Fig. 7.1).

> "75. Wir verstehen durch die Wirksamkeit einer Kraft die Integral-Grösse, welche gefunden wird, wenn man die Kraft mit dem Differentiale ihrer Entfernung von der Fläche, von welcher sie den Körper wegstösst, multiplicirt und alsdann integriert." [Euler E842, § 75]

Then, Euler referred to Maupertuis' principle of rest.

> "Denn es kann bewiesen werden, dass kein Gleichgewicht stattfinden kann, wo nicht die Summe der Wirksamkeiten aller Kräfte, so dabei vorkommen, am allerkleinsten oder auch zuweilen am allergrössten ist." [Euler E842, § 75][16]

[16] Euler's papers on the same subject are the following: [Euler E145], [Euler E146], [Euler E176], [Euler E181], [Euler E182], [Euler E186], [Euler E197], [Euler E198], [Euler E199], [Euler E200].



$$2\,M\,udu = -V\,dv - V'\,dv' - V''\,dv''\ \&c.$$

on aura en prenant les intégrales

$$M\,uu = \text{Conſt.} - \int V\,dv - \int V'\,dv' - \int V''\,dv'' - \&c.$$

XVIII. Donc, puisque par l'hypotheſe $\int V\,dv + \int V'\,dv' + \int V''\,dv''$ &c. exprime l'effort des forces ſur le corps M, que j'ai poſé $= \Phi$, il eſt évident que nous aurons : $M\,uu = \text{Conſt.} - \Phi$.

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Fig. 7.1 Euler's paper on Maupertuis' principle. The formula for the sum of "living forces" and "efficiency" is related to an integration constant [Euler E197]

The representation of the Wirksamkeit (effort) had been also given in the paper *Harmonie entre les principes généraux de repos et de mouvement de M. de Maupertuis* [Euler E197].

> "(...) und kann hernach für eine jegliche Zeit und für einen jeglichen Ort des Körpers seine wahre lebendige Kraft richtig bestimmt werden. Dieses ist der beträchtliche Vorzug, welchen das Product der Masse eines Körpers durch das Quadrat seiner Geschwindigkeit vor dem Product der Masse durch die Geschwindigkeit selbst hat und [der] den Begriff der lebendigen Kraft über die Grösse der Bewegung weit erhebet ...)." [Euler E842, § 75]

Methodologically, the laws of mechanics are not only experimentally confirmed, but had been related to a reliable conceptual frame awarding to them the same status of necessity the basic relations of geometry have, or, following Euler, "aller de pair avec les vérités géométriques" [Euler E344, Lettre LXXI]. Euler referred to the subdivision into *necessary* and *contingent* truths which was not only commonly accepted in the first half of the 18th century as a methodological principle, but had been also used later. Using the concept of function, the relations obtained for constant forces are readily generalized for coordinate and time dependent forces if coordinate and time dependence are treated separately.

The Eqs. (7.16) and (7.17) describe the motion of a mass point and are generally valid for all types of forces which are independent of velocity. These forces had been later called "Bewegungskräfte" by Helmholtz [Helmholtz, Vorlesungen]. The coordinate dependent forces had been called "conservative forces".

> Die letztgenannten Gleichungen leiten uns auf einen neuen Begriff, welchen wir mit dem Worte Wirksamkeit verbinden; indem aus demselben durch die Integration gefunden wird $Muu = \int dxP$, $Mvv = \int dyQ$, $Mww = \int dzR$, und hieraus die Bewegung des Körpers von einer jeden Fläche angezeigt werden kann. Dieser Begriff ist hier um so viel mehr von der größten Wichtigkeit, weil die Summe der Wirksamkeit aller Kräfte $\int Pdx + \int Qdy + \int Rdz$ immer gleich groß bleibt, wenn wir auch drei andere Flächen angenommen hätten; (...). Ein solcher merkwürdiger Vorzug findet bei der anderen Formel nicht statt und würde zum Exempel die aus der ersten hergenommene Größe $\int Pdt + \int Qdt + \int Rdt$ immer einen anderen Wert erhalten, je nachdem wir die drei Flächen veränderten. [Euler E842, § 75]

Although Euler derived the expression $Muu = \text{Const} - \Phi$ [Euler E197, § 18] between *living forces* and *Wirksamkeit*, nothing else than the total energy as the sum of kinetic and potential energy, he did not pay attention to the constant $Muu + \Phi =$ Const made up by the sum of the two terms, but discussed the properties of the terms in view of their maxima and minima. In the 19th century, Helmholtz made use of the same relations to demonstrate the conservation of energy and interpreted the integration constant as total energy.

Chapter 8
Euler's Mechanics and Schrödinger's Quantum Mechanics

In 1900, quantum mechanics had been invented by Planck who made use of the 2nd Law of thermodynamics to explain the observed behaviour of black body radiation. In the end of the 19th century, Newtonian mechanics had been completed by Maxwell who introduced the theory of electromagnetisms [Maxwell, Electromagnetism] in 1865. The light velocity turned out to be an inherent component of the Maxwell's theory. Planck made use of both theories and constructed a system of classical harmonic oscillators which are interacting with an external radiation field described by Maxwell's theory.[1] Immediately after the experimental findings of Wien and Rayleigh and Jeans had been explained by the invented unified approach, Planck began to think about the relation of his new concept to classical mechanics. Planck was fully aware of the decisive step beyond the frame of classical mechanics and electrodynamics whereas the physics community did not respond until 1905.[2] Planck referred to the basic assumption made by Newton and

[1] "The subject of quantum physics started with the statistical theory of the distribution of energy in the black-body spectrum. The spectrum of radiated energy in equilibrium with matter in an enclosure is commonly called black-body radiation because it is the kind of radiation that would be emitted by a perfect absorber. The active problem in 1900 was the explanation of the distribution of energy in the spectrum. It is interesting to realize that the subject has quite an ancient history. The first application of thermodynamics to black-body radiation goes back to 1859 when Kirchhoff first developed the ideas of radiative exchanges, and the connections between emission and absorption, rules according to which a good emitter is a good absorber, and a poor emitter is a poor absorber. In 1884 the discovery had been made of what we now call the Stefan-Boltzmann law, that the total radiation goes up as the fourth power of the absolute temperature. It was discovered by Stefan experimentally and interpreted theoretically by Boltzmann, making one of the earliest applications of thermodynamics to radiation after those first ideas of Kirchhoff's." [Condon]

[2] "Thus in the space of just a month or two Planck first found an empirical formula which to this day gives the most accurate representation of the spectral distribution of the radiant energy; second, he found a derivation of that formula. In order to get the derivation he had to introduce the extraordinary idea of energy quantization into physics. Third, he obtained an excellent value for the charge on the electron, which everybody was trying to do at that time.

You might expect that this would cause a great deal of excitement among physicists at that time, but it did not. If you search through the journals you find that practically nothing is said about Planck in the years 1900 through 1904. I was very much intrigued, therefore, when just before this meeting Mr. Marton recalled that a search of the records of this Society indicated that in 1902 Arthur L. Day gave a report on Planck's work. Thus The Philosophical Society of Washington was one of the earliest to pay attention to it." [Condon]

D. Suisky, *Euler as Physicist*, DOI 10.1007/978-3-540-74865-6_8,

Leibniz on the continuity of all processes in nature which had been expressed by the calculus. Moreover, Planck emphasized the gulf between the disciplines and guessed that there is no hope for reconciliation.

> Das Scheitern aller Versuche, diese Kluft zu überbrücken, ließ bald keinen Zweifel mehr daran übrig, daß das Wirkungsquantum in der Atomphysik eine fundamentale Rolle spielt und daß mit seinem Auftreten eine neue Epoche in der physikalischen Wissenschaft anhebt. Denn in ihm kündet sich etwas bis dahin Unerhörtes an, das berufen ist, unser physikalisches Denken, welches seit der Begründung der Infinitesimalrechnung durch Leibniz und Newton sich auf der Annahme der Stetigkeit aller kausalen Zusammenhänge aufbaut, von Grund aus umzugestalten. [Planck SelbstBiographie][3]

Planck intended to "integrate the action parameter" into the frame of classical mechanics which is based on the assumption that "all causal connections are continuous". Planck acknowledged the merits of Bohr and Heisenberg who explained the relation between classical and quantum mechanics by the *correspondence principle* whereas Schrödinger constructed a differential equation thereby generating the wave corpuscle dualism.[4]

Einstein was the first who comprehended the importance of Planck's discovery and presented the hypothesis of light quanta [Einstein, Heuristisch], [Einstein, Absorption] Planck commented:

> I therefore tried immediately to weld the elementary quantum of action, h, somehow into the framework of classical theory. But in the face of all such attempts the constant showed itself to be obdurate. [Condon][5]

Quantum mechanics is known to be essential different from classical mechanics, but, nevertheless, in need of a foundation by classical mechanics. The relation between classical and quantum mechanics had been considered for the first time by

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[3] "(...) for it heralded the advent of something entirely unprecedented and was destined to remodel basically the physical outlook and thinking of man of (...) our physical thinking which, since the foundation of the infinitesimal calculus by Leibniz and Newton, was founded on the assumption of the continuity of all causal connections." [Condon]

[4] "Meine vergeblichen Versuche, das Wirkungsquantum irgendwie der klassischen Theorie einzugliedern, erstreckten sich auf eine Reihe von Jahren und kosteten mich viel Arbeit. Manche Fachgenossen haben darin eine Art Tragik erblickt. Ich bin darüber anderer Meinung. Denn für mich war der Gewinn, den ich durch solch gründliche Aufklärung davontrug, um so wertvoller. Nun wußte ich ja genau, daß das Wirkungsquantum in der Physik eine viel bedeutendere Rolle spielt, als ich anfangs geneigt war anzunehmen, und gewann dadurch ein volles Verständnis für die Notwendigkeit der Einführung ganz neuer Betrachtungs- und Rechnungsmethoden bei der Behandlung atomistischer Probleme. Der Ausbildung solcher Methoden, bei der ich selber nun allerdings nicht mehr mitwirken konnte, dienten vor allem die Arbeiten von Niels Bohr und von Erwin Schrödinger. Ersterer legte mit seinem Atommodell und mit seinem Korrespondenzprinzip den Grund zu einer sinngemäßen Verknüpfung der Quantentheorie mit der klassischen Theorie. Letzterer schuf durch seine Differentialgleichung die Wellenmechanik und damit den Dualismus zwischen Welle und Korpuskel." [Planck SelbstBiographie]

[5] "Wenn nun die Bedeutung des Wirkungsquantums für den Zusammenhang zwischen Entropie und Wahrscheinlichkeit endgültig feststand, so blieb doch die Frage nach der Rolle, welche diese neue Konstante bei dem gesetzlichen Ablauf der physikalischen Vorgänge spielt, noch vollständig ungeklärt. Darum bemühte ich mich alsbald, das Wirkungsquantum h irgendwie in den Rahmen der klassischen Theorie einzuspannen. Aber allen solchen Versuchen gegenüber erwies sich diese Größe als sperrig und widerspenstig." [Planck, SelbstBiographie]

Planck in 1900 who introduced the quantum of action in the theory of black body radiation [Planck, Vorlesungen], [Planck 1906], [Planck 1913 (Masius)]. In 1906 Einstein based the theory of specific heat on Planck's assumption [Einstein, Wärme] and, in 1913, Bohr introduced Planck's parameter into the theory of atomic spectra [Bohr 1 to 4].

Heisenberg [Heisenberg 1925], [Heisenberg B J 1926] and Schrödinger [Schrödinger, First to Fourth Announcement] started from the following assumptions: (i) the existence of Planck's action parameter, (ii) the classical Newtonian equation of motion [Heisenberg 1925] or a classical wave equation [Schrödinger, Second Announcement] and, (iii) the energy conservation for stationary states. Additionally, Heisenberg rejected the concept of paths (iv) and Schrödinger assumed the existence of a wave function being the solution of an eigenvalue problem (v).

8.1 The Historical Background of the Development of Quantum Mechanics

In 1900 and 1913, Max Planck, although having destroyed the traditional representation of mechanics based on the principle of continuity by the invention of the action parameter, constructed a theory for the interaction of a black body with an external radiation field which is based on the simultaneously assumed processes of (i) absorption and (ii) emission of radiation being explained in terms of a *continuous absorption* and *discontinuous emission* of radiation, respectively. Planck defined elementary areas in the phase space depending on *coordinates* and *momentums* of the oscillators [Planck, Vorlesungen]. In 1906, Einstein invented a non-geometrical model based on the series of *discrete energetical* states of each of these oscillators [Einstein, Wärme][6] which are represented neither in terms of coordinate nor momentum nor time dependent functions. Einstein reduced and reformulated the problem in terms of a relation between a *continuum* and a *discrete* set of numbers. The "arithmetization" of the problem is obvious since the energetic states are represented by numbers, moreover, integers. The problem of *quantization* can be understood as a procedure for the *selection* of this discrete set of energies from the underlying classical continuum. Therefore, it had been assumed that the *internal* states of the parts the system is made up are described by *total energy* of the oscillator. Einstein assumed that the energetic density is not described by a continuous function, but by a function which is composed of *finite* and an *infinitesimal* terms denote by $n \cdot \varepsilon$ and α, respectively.

> Statt daß wir indessen gemäß der molekular-kinetischen Theorie $\omega(E) = \text{const}$ setzen, setzen wir $\omega = 0$ für alle Werte der Energie E, welche den Werten $0, \varepsilon, 2\varepsilon, 3\varepsilon$ etc. nicht

[6] "Über das Gesetz, nach dem sich die betrachteten Punkte in dem Raume bewegen, sei nichts vorausgesetzt, als daß in bezug auf diese Bewegung kein Raumteil (und keine Richtung) von der anderen ausgezeichnet sei." [Einstein, Heuristisch]

> außerordentlich nahe liegen. Nur zwischen 0 und $0+\alpha$, ε und $\varepsilon+\alpha$, 2ε und $2\varepsilon+\alpha$ etc. (wobei α unendlich klein sei gegen ε) sei ω von Null verschieden, derart, daß
>
> $$\int_0^\alpha \omega dE = \int_\varepsilon^{\varepsilon+\alpha} \omega dE = \int_{2\varepsilon}^{2\varepsilon+\alpha} \omega dE = \ldots = A$$
>
> sei. Diese Festsetzung involviert (...) die Annahme, daß die Energie des betrachteten Elementargebildes lediglich solche Werte annehme, die den Werten $0, \varepsilon, 2\varepsilon, 3\varepsilon$ etc. unendlich nahe liegen. [Einstein, Wärme]

In case of Planck's theory the elements of the system are non-interacting linear harmonic oscillators. The interaction is only between the oscillator and the light waves of an external radiation field. Following Planck, the internal energy is assumed to be *continuously* increasing until a certain *finite* critical value represented by a multiple of $\varepsilon = h \cdot \nu$, i.e. $\varepsilon_n = n \cdot h \cdot \nu$, where the emission *instantaneously* sets in [Planck, Vorlesungen, §§ 147 and 150]. Later, the assumption on the states had been slightly, but seriously in the consequences, modified by replacing the *integers* with *rational* numbers. Hence, the state $\varepsilon_0 = 0$ is *excluded* and the quantum mechanical representation is not only distinguished by the *arithmetically defined criterion* of the difference in the *number* of states. Einstein assumed that the number of states of a molecular system is less than the number of states observed for bodies of our experience.

> Aus dem Vorhergehenden geht klar hervor, in welchem Sinne die molekular-kinetische Theorie der Wärme modifiziert werden muß, um mit dem Verteilungsgesetz der schwarzen Strahlung in Einklang gebracht zu werden. Während man sich nämlich bisher die molekularen Bewegungen genau denselben Gesetzmäßigkeiten unterworfen dachte, welche für die Bewegungen der Körper unserer Sinnenwelt gelten, (...), sind wir nun genötigt (...), die Annahme zu machen, daß die Mannigfaltigkeit der Zustände, welche sie anzunehmen vermögen, eine geringere sei als bei den Körpern unserer Erfahrung. [Einstein, Wärme]

Hence, the interpretation of quantum mechanics in terms of "arithmetization" is to be related to the corresponding interpretation of classical mechanics in terms of "arithmetization". The difference between classical and quantum mechanics as well as their relations are to be traced back (i) to physical principles and (ii) mathematical relations between different sets of numbers.

Following Einstein, the physical principles are to be traced back to energy conservation and the symmetry between *absorption* and *emission* [Einstein, Heuristisch].[7]

> Die Energie eines Elementarresonators kann nur Werte annehmen, die ein ganzzahliges Vielfaches (...) sind, die Energie eines Resonators ändert sich durch Absorption und Emission sprungweise (...). [Einstein, Heuristisch]

The mathematical principles are to be traced back to appropriate *functions* of *coordinates* and *time* where the former ones describe the extension of the system in

[7] "Nach (...) der Annahme ist (...) die Energie nicht kontinuierlich verteilt, sondern es besteht dieselbe aus einer endlichen Zahl von in Raumpunkten lokalisierten Energiequanten, welche sich bewegen, ohne sich zu teilen und nur als Ganzes absorbiert und erzeugt werden." [Einstein, Heuristisch]

dependence on the energy accumulated in the system. Obviously, the functions are also to be physically interpreted.[8]

8.2 Planck on Newton and Leibniz

Planck intended to join the classical model of continuous changes which describes the absorption of energy according to Newton's and Maxwell's theory and the discontinuous emission of light which is incompatible with these classical models. After the invention of the action parameter in 1900, Planck presented different version of foundation to make the theory compatible with the principles of classical mechanics. Planck formulated the problem in terms of completeness of the theory and the hypothesis of hidden processes behind the discontinuous changes of the states. In 1912, Planck described the procedure.

> Da nun in dem angenommenen Absorptionsgesetz die Quantenhypothese noch keinen Platz gefunden hat, so folgt, daß sie bei der Emission der Oszillatoren irgendwie zur Geltung kommen muß, und dies geschieht durch die Einführung der Hypothese der *Quantenemission*. Wir wollen nämlich voraussetzen, daß die Emission nicht, wie Absorption, kontinuierlich erfolgt, sondern daß sie nur in ganz bestimmten Zeitpunkten, plötzlich, stoßweise, einsetzt, und zwar nehmen wir speziell an, daß ein Oszillator nur in einem solchen Zeitpunkt Energie emittieren kann, in dem seine Schwingungsenergie gerade ein ganzes Vielfaches n des Energiequantums $\varepsilon = h\nu$ geworden ist. Ob er dann wirklich emittiert, oder ob seine Schwingungsenergie noch weiter durch Absorption zunimmt, soll vom Zufall abhängen. Nicht als ob für die Emission keine Kausalität angenommen würde; aber die Vorgänge, welche die Emission kausal bedingen, sollen so verborgener Natur sein, daß ihre Gesetze einstweilen nicht anders als auf statistischen Wege ermittelt werden können. [Planck, Vorlesungen, § 147]

Hence, Planck indirectly assumed that mechanics may be considered either as an *incomplete* theory or as a theory which can be generalized such that discontinuous processes can be taken into account. Nevertheless, the significant modification of basic principles is evident.

> While the significance of the quantum of action for the interrelation between entropy and probability was thus conclusively established, the great part played by this new constant in the uniform regular occurrence of physical processes still remained an open question. I therefore tried immediately to weld the elementary quantum of action, h, somehow into the framework of classical theory. But in the face of all such attempts the constant showed itself to be obdurate. [Condon 1962]

The crucial point is the interpretation of continuity. Following Newton and Leibniz, continuity is observed in nature preferentially in the motion of bodies. Leibniz explicitly excluded leaps from the world made up of bodies. After the invention of

[8] "(...) es ist wohl denkbar, daß die mit kontinuierlichen Raumfunktionen operierende Theorie des Lichts zu Widersprüchen mit der Erfahrung führt, wenn man sie auf die Erscheinungen der Lichterzeugung und Lichtverwandlung anwendet." [Einstein, Heuristisch]

the calculus, the representation of motion and the change of motion had been formulated in terms of differentials, differentio-differentials, fluents and fluxions (compare Chaps. 2 and 3) and relations between these quantities. Later, the differential equations had been interpreted in terms of causality and causal interaction regarded as continuous. However, in Euler's model, the path is composed of those parts where the bodies are not interacting and those where the bodies are interacting. The change from free motion to interaction and from interaction to free motion is *discontinuous* although the *necessity* in the world of bodies is not modified by these successions of events of different kind. The *causal connection* between all parts of motion and interaction is always preserved.[9]

> So long as it could be regarded as infinitesimally small, i.e., dealing with higher energies and longer periods of time, everything was in perfect order. But in the general case difficulties would arise at one point or another, difficulties which became more noticeable as higher frequencies were taken into consideration. The failure of every attempt to bridge that obstacle soon made it evident that the elementary quantum of action plays a fundamental part in atomic physics and that its introduction opened up a new era in natural science, for it heralded the advent of something entirely unprecedented and was destined to remodel basically the physical outlook and thinking of man which, ever since Leibniz and Newton laid the ground work for infinitesimal calculus, were founded on the assumption that that all causal interactions are continuous. [Condon 1962]

The relation between continuity and discontinuity can be mathematically modelled by the calculus of differences [Euler E 212] (compare Chap. 5).

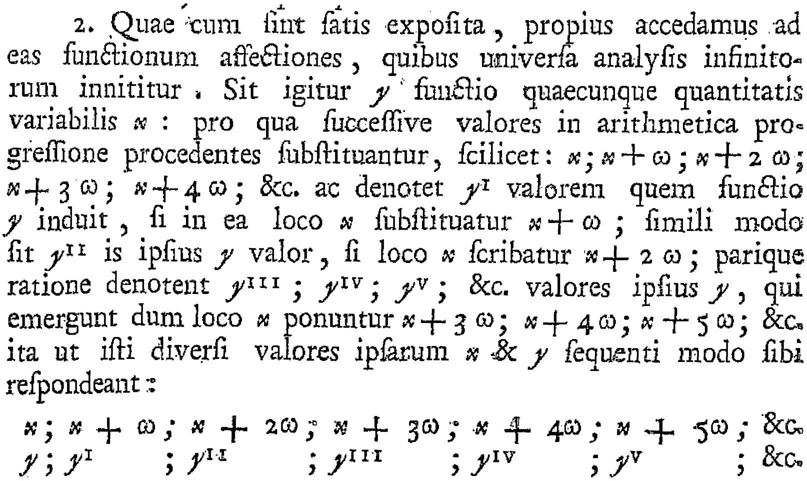
8.3 Discrete and Continuous Quantities

8.3.1 Discrete and Continuous Variables in the Calculus of Differences

Following Euler, the basic laws of mechanics are "not only true, but necessarily true" [Euler E016/016, § 131] and, therefore, all consequences which are deductively obtained have the same status.[10] Moreover, following Euler, there are two kinds of discontinuity mathematically represented by the calculus of differences and the differential calculus (compare Chap. 5). In both cases, the independent variables are represented in terms of *discrete arithmetical progressions*. Hence, follow-

[9] This requirement had been also formulated by Planck. "Denn in ihm kündigt sich etwas bis dahin Unerhörtes an, das berufen ist, unser physikalisches Denken, welches seit der Begründung der Infinitesimalrechnung durch Leibniz und Newton auf der Annahme der Stetigkeit aller kausalen Zusammenhänge aufbaut, von Grund auf umzugestalten." [Planck, SelbstBiograhie]

[10] This feature had been later called "determinisms", i.e. there are no exclusions in the prediction of the paths or parts of the path, the path of a body is either determinate in all details and for all times [Euler E842, § 21] or it does not exist at all. "21. *Zu einer vollständigen Erkenntniss aber der Bewegung eines Punktes wird nicht nur erfordert, dass man den von demselben beschriebenen Weg anzuzeigen wisse, sondern man muss in diesem Wege für einen jeglichen Zeitpunkt die Stelle bestimmen können, wo sich der bewegte Punkt damals befunden*." [Euler E842, § 21]



2. Quae cum ſint ſatis expoſita, propius accedamus ad eas functionum affectiones, quibus univerſa analyſis infinitorum innititur. Sit igitur y functio quaecunque quantitatis variabilis x: pro qua ſucceſſive valores in arithmetica progreſſione procedentes ſubſtituantur, ſcilicet: $x; x+\omega; x+2\omega; x+3\omega; x+4\omega$; &c. ac denotet y^{I} valorem quem functio y induit, ſi in ea loco x ſubſtituatur $x+\omega$; ſimili modo ſit y^{II} is ipſius y valor, ſi loco x ſcribatur $x+2\omega$; parique ratione denotent y^{III}; y^{IV}; y^{V}; &c. valores ipſius y, qui emergunt dum loco x ponuntur $x+3\omega$; $x+4\omega$; $x+5\omega$; &c. ita ut iſti diverſi valores ipſarum x & y ſequenti modo ſibi reſpondeant:

$x;\ x+\omega;\ x+2\omega;\ x+3\omega;\ x+4\omega;\ x+5\omega;$ &c.
$y;\ y^{I};\ y^{II};\ y^{III};\ y^{IV};\ y^{V};$ &c.

Fig. 8.1 Euler's calculus of differences [Euler E212, § 2]

ing Euler, the foundation of the calculus of differences is an *inherently discontinuous* theory, but simultaneously also valid for *continuous* variables whose *discrete increments* are investigated (see Fig. 8.1).

The independent variable x is represented by a *series of terms* generated by *multiples* of the *increments* ω which are correlated to a second *series* of terms made up by the values of the *function* belonging to different arguments of the continuous variable x. Since the arithmetical series can be continued to infinity, the series of the values of the function can be also carried onto infinity.

> 3. Quemadmodum series arithmetica x; $x+\omega$; $x+2\omega$; & $c\ldots$. in infinitum continuari potest, ita series ex functione y; y^{I}; y^{II}; &c. quoque in infinitum progreditur, iusque natura pendebit ab indole functionis y. [E212, § 3][11]

This scheme is completely preserved for infinitesimal increments ε which, maintaining the continuous variable x, substitutes the finite increment ω (see Tables 8.1 and 8.2).

Table 8.1 Euler's calculus of finite increments [Euler E212, Chap. 1]. The superscript indicates different values of functions (compare Chap. 5)

x;	$x+\omega$;	$x+2\omega$;	$x+3\omega$;	$x+4\omega$;...
y;	y^{I};	y^{II};	y^{III};	y^{IV};...

Obviously, the independent variable is also represented by a discrete series as before. The only difference is due to the magnitude of the increments, but not due

[11] Euler accentuated the "innate character or inborn quality of the function y", i.e. the inherent properties of the function independent of the magnitude of the argument and the increment.

Table 8.2 Euler's calculus of infinitesimal increments. The superscript indicates different values of functions (compare Chap. 5). The continuity does not follows from the calculus, but is due to the continuity of the variable x and the function $y = y(x)$

x;	$x+\varepsilon$;	$x+2\varepsilon$;	$x+3\varepsilon$;	$x+4\varepsilon;\ldots$
y;	y^I;	y^{II};	y^{III};	$y^{IV};\ldots$

to the magnitude of the variable x. Interpreting the increments mechanically either as temporal or spatial intervals, motion is described in the frame of two models, first, the *Galilean* model of finite intervals and finite increments (compare Chap. 1) or the *Newtnian* model of a continuous flux (compare Chaps. 2 and 3). Nevertheless, making use of Euler's representation [Euler E212] the flux is resolved into an arithmetical progression of infinitesimal temporal intervals. Supposing first and last ratios of fluxions [Newton, Principia, Book I, Lemma I] (compare Chap. 3), Newton confined the validity of the model to those expressions generated by the first or last increments. [12]

8.3.2 Discrete Series of Energies

Following Planck, the arithmetical progression is constituted by different energies of the harmonic oscillators the system consists of. These oscillators are independent of each other, i.e. their internal states depend only on internal parameters. The absorption and emission of energy is also not mechanically correlated, but thermodynamically described by the hypothesis of elementary disorder. The energies each of the oscillators can accumulate are described by an arithmetical series. Furthermore, it is assumed that there are functions belonging to each of the states. The only difference in comparison to the previous examples is that the functions are not functions of the energy, but functions of *coordinates* or *time* (see Tables 8.3 and 8.4).

Table 8.3 Energetic states described by arithmetic series of energies and functions of coordinates. The subscript indicates different states of the system[12]

States of the system, the increments $n \cdot \Delta E$ are of finite magnitude
$E_0;\ E_0+\Delta E;\ E_0+2\Delta E;\ E_0+3\Delta E;\ E_0+4\Delta E;\ldots$
Functions belonging to the states of the systems
$y_0(x);\ y_1(x);\ y_2(x);\ y_3(x);\ y_4(x);\ldots$

[12] The arithmetical series of energetic states had been introduced for the first time by Einstein [Einstein, Wärme]. "(...) sind wir nun genötigt (...), die Annahme zu machen, daß die Mannigfaltigkeit der Zustände, welche sie anzunehmen vermögen, eine geringere sei als bei den Körpern unserer Erfahrung." [Einstein, Wärme] As later Heisenberg [Heisenberg 1925], Einstein exclusively referred to *experimental data* for the construction of the model and concluded that molecular systems reveal properties which are essentially different from those of bodies known before.

Table 8.4 Energetic states described by arithmetic series of energies and functions of time. The subscript indicates different states of the system

States of the system, the increments $n \cdot \Delta E$ are of finite magnitude
$E_0;\ E_0 + \Delta E;\ E_0 + 2\Delta E;\ E_0 + 3\Delta E;\ E_0 + 4\Delta E; \ldots$
Functions belonging to the states of the systems
$r_0(t);\ r_1(t);\ r_2(t);\ r_3(t);\ r_4(t); \ldots$

Alternatively, the states can be related to time dependent functions.

The reason is that E is not a variable like coordinates and time, but a parameter of the system, e.g. $E = E_0 = \text{const}$. Furthermore, all increments $n \cdot \Delta E$ are of *finite* magnitude and cannot be replaced with infinitesimal increments of energy. Hence, the calculus can only be applied provided that there are other functions which can be treated in Euler's calculus of differentials. The only candidates currently known from Euler's mechanics are functions of *coordinates* $y(x)$ or functions of *time* $r(t)$ whose shape, however, is unknown until now. Following Euler, the equations of motions are formulated in terms of infinitesimal increments of velocity dv which are either proportional to elements or differentials of space (spatiolum) dx or elements or differentials of time (tempusculum) dt [Euler E015/016, §§ 150–154].[13]

Each oscillator is described by the arithmetical series of energies representing the different states of the system. As a consequence, there are no other variables except coordinates and time and the functions belonging to each of the states are either functions of coordinates or functions of time. Furthermore, it had been assumed that the states of the system can be described either by purely time dependent or purely coordinate dependent functions. Functions of such type had been introduced by Heisenberg [Heisenberg 1925] (time dependent functions) and Schrödinger [Schrödinger, Second Announcement] (coordinate dependent functions, amplitude equation). The superscript does not indicate a *difference* between two values of the function as before, but the *state* the functions belongs to. Therefore, the superscript is replaced with a subscript to avoid a mixing up.

The series of states had been experimentally investigated. Planck established for the first time a relation between the *discrete* action and a *discretization* of states of molecular systems [Planck, Vorlesungen]. The discrete states are described by the division of the phase space into a multitude of regions of finite extension.[14] By this procedure, Planck obtained a consistent theory making use of Boltzmann's postulate on the correlation between entropy and probability and removed the inconsistency caused by the indeterminate absolute value of entropy in classical thermodynamics [Planck, Vorlesungen, § 120]. Using Planck's relation between energy and time,

[13] In the frame of the calculus, the *independence* of space and time as physical quantities is expressed in terms of the distinguished differentials dx and dt which cannot be reduced to each other. Then, the time dependent change of the velocity can be distinguished from the coordinate dependent change and is represented by different equations.

[14] The interpretation of these regions as being caused by the discrete states of oscillators is due to Einstein [Einstein, Wärme].

Table 8.5 Functions related to the amplitude equation [Schrödinger, Second Announcement]

States of the system, the increments $n \cdot \Delta E$ are of finite magnitude
$E_0;\ E_0+\Delta E;\ E_0+2\Delta E;\ E_0+3\Delta E;\ E_0+4\Delta E;\ldots$
Amplitude functions belonging to the states of the systems
$\psi_0(x);\ \psi_1(x);\ \psi_2(x);\ \psi_3(x);\ \psi_4(x);\ldots$

$E = \hbar \cdot \omega$, the states can be alternatively described in terms of arithmetic series of frequencies [Schrödinger, Second Announcement, p. 513].[15] The difference to classical mechanics is that the states are not only characterized by invariant parameters like velocity and energy, but by coordinate and time dependent functions.

Schrödinger proposed to correlate the discrete vibrations or energies with purely coordinate dependent functions (compare Table 8.3) which are defined in the whole configuration space [Schrödinger, Second Announcement],[16] [Schrödinger, Heisenberg].[17] Hence, following Schrödinger, the scheme is to be completed by the wave function or amplitude functions where the latter are related to the time-independent amplitude equation [Schrödinger, Fourth Announcement] (see Table 8.5). Assuming a model system, the linear harmonic oscillator, the relation between the states of the system and the proposed coordinate dependent functions can be specified as solutions of a differential equation [Schrödinger, Second Announcement].

Interpreting Planck's theory and considering experimental data, Einstein constructed a model for the energetic states of the harmonic oscillator [Einstein, Wärme] in 1906. Einstein assumed $E_0 = 0$, then, the series of states are

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[15] "Die geäußerte Befürchtung verwandelt sich also in ihr Gegenteil, jedenfalls was die Energieniveaus oder sagen wir vorsichtiger, die Frequenzen betrifft. (Denn was es mit der 'Energie der Schwingungen' auf sich hat, ist eine Frage für sich, (...))." [Schrödinger, Second Announcement, p. 513]

[16] "Denn es zeigt sich (...), daß die Gleichung (...) *die Quantenbedingungen in sich trägt*. Sie sondert in gewissen Fällen (...) die für stationäre Vorgänge allein möglichen aus, ohne irgendeine weitere Zusatzannahme als die für eine physikalische Größe beinahe selbstverständliche Anforderung an die Funktion Ψ: dieselbe soll im ganzen Konfigurationenraum eindeutig endlich und stetig sein." [Schrödinger, Second Announcement, p. 511] Hence, Schrödinger interpreted the assumption in the frame of the classical picture where the invariance or time independence of certain physical quantities, like energy or momentum, is related to changes within the investigated system performed by the parts of the system (compare Helmholtz's analysis of the conservation of living forces where the bodies periodically return to their initial positions after a certain time [Helmholtz, Vorlesungen, § 48]). This interpretation follows from Schrödinger's intention to replace the usual mechanics with an *undulatory* mechanics [Schrödinger, Second Announcement, pp. 496, 497, 506–509], [Schrödinger, Nobel Lecture]. "Aber schon der erste Versuch einer wellentheoretischen Ausgestaltung führt auf so frappante Dinge, daß ein ganz anderer Verdacht aufsteigt: wir wissen doch heute, daß unsere klassische Mechanik bei sehr kleinen Bahndimensionen (...) versagt. (...) Dann gilt es, eine 'undulatorische Mechanik' zu suchen (Vgl. A. Einstein, Berl. Ber. S. 9ff. 1925)." [Schrödinger, Second Announcement, pp. 496–497]

[17] "(N.B. nicht '*pq-Raum*', sondern '*q-Raum*'.)" [Schrödinger, Heisenberg, p. 736]

given by 0, ε, 2ε, $3\varepsilon, \ldots$. [Einstein, Wärme, p. 182]. Einstein's assumption can be directly related to Planck's theory of black body radiation. Planck derived a relation between energy and entropy of the system [Planck, Vorlesungen, §§ 139–140]. The parameters and variables in Planck's formula are energy (the product of action parameter and the frequency of the oscillator), the action parameter and the temperature. Kinetic and potential energy are never separately considered. Hence, the energy parameter are neither composed of kinetic and potential energies nor split into such components.

Heisenberg and Schrödinger generalized Einstein's model and asked for a decomposition of the total energy of oscillators into the usually accepted components of kinetic and potential energies. Then, the total energy is not an experimentally determinate parameter, but has to be derived from additional assumptions. Both authors assumed that the total energy is composed of the parts known from classical mechanics. As a consequence, the discrete states of a quantum mechanical system can be described by coordinate dependent functions which follows from their dependence oh the potential energy $V(x)$.

In case of harmonic oscillator, the increment of energy ΔE is *experimentally* obtained and, consequently, a determinate parameter whose value can be modified only by additional suppositions, e.g. the inclusion of anharmonicity [Heisenberg 1925]. Following Planck, the emitted energy is given by the relation $\Delta E = \hbar \cdot \omega$ [Planck, Vorlesungen, §§ 125, 135, 145, 147].[18] The frequency of the oscillator depends on the material parameters of the oscillator, i.e. its mass m and force constant k, whose magnitudes are independent of absorption and emission processes, $\omega = \sqrt{k/m}$. Hence, the only *indeterminate* quantity is the parameter E_0. From Table 8.5, it follows that all functions assumed for the description of the system depend on the same parameter since it holds $\Delta E = \hbar\omega$.

Following Einstein [Einstein, Wärme, p. 183] and assuming, that the "number of energetic states" of a molecular system should be less than those of the "bodies of our sensual experience", it is straightforwardly that, for one and the same system, the possible values of parameter E_0 cannot form a continuum, but are to be chosen from a countable set of possible values, at least there is one and only one value of the parameter. Then, it can be expected that there is also one and only one function $\psi_0(x)$ belonging to this state. Hence, Einstein formulated a *selection* problem[19] which had been later interpreted in terms of an *eigenvalue* problem by Schrödinger [Schrödinger, First to Fourth Announcement].

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[18] "Durch die vorstehenden Entwicklungen ist die Berechnung der Entropie eines Systems von N Molekülen in einem gegebenen thermodynamischen Zustand ganz allgemein zurückgeführt auf die einzige Aufgabe, die Größe G der Elementargebiete des Zustandsraumes aufzufinden. Daß eine derartige ganz bestimmte endliche Größe wirklich existiert, ist der hier entwickelten Theorie eigentümlich und bildet den Inhalt der sogenannten *Quantenhypothese*. Dieselbe ist, wie man sieht, eine unmittelbare Folgerung aus dem Satz des § 120, daß der Entropie ein absoluter Wert zukommt. (...)." [Planck, Vorlesungen, §§ 125 and 147]

[19] The system is not described by the motion of its parts, but only by the material parameters and the action parameter. The magnitude of the action parameter is independent of the system [Planck, Vorlesungen, § 162].

Following Einstein, the energetic states belong to the same system and are characterized by the "same weight". The result of integration is independent of the state, i.e. the constant A is assumed to be independent of the energy.

$$\int_0^{\alpha} \omega(E)dE = \int_{\varepsilon}^{\varepsilon+\alpha} \omega(E)dE = \int_{2\varepsilon}^{2\varepsilon+\alpha} \omega(E)dE = \ldots = A \tag{8.1}$$

Einstein's representation can be reformulated in terms of Schrödinger's coordinate dependent wave functions which are related to different states of the system.

$$\int_{-\infty}^{+\infty} dx\,|f_1(x)|^2 = \int_{-\infty}^{+\infty} dx\,|f_2(x)|^2 = \int_{-\infty}^{+\infty} dx\,|f_3(x)|^2 = \ldots = B \tag{8.2}$$

where the functions $f_1(x) \neq f_2(x) \neq f_3(x) \neq \ldots$ are different from each other since they belong to different energetic states of the system. The dependence on coordinates is directly related to the energy of the system depending on the state. As it had been discussed by Helmholtz, the extension of the system is correlated to the total energy (compare Chap. 7). Now, Helmholtz's idea had been generalized and the "extension" is described by a coordinate dependent function which is related to *all* configurations of the system. The infinite set of different functions $f_1(x), f_2(x), f_3(x), \ldots$, belonging to the same system, describe the dependence of extension on the energy. As in Euler's mechanics, the change of the state is only due to an external cause $V_{\text{ext}}(x)$, i.e.

$$\int_{-\infty}^{+\infty} dx \cdot f_{\text{initial}}(x) \cdot f_{\text{final}}(x) \cdot V_{\text{ext}}(x) \neq 0. \tag{8.3}$$

But, in contrast to Euler's mechanics,[20] the change of the state is *not independent* of the initial state since the result depends on the function of initial state $f_{\text{initial}}(x)$ as well as on the function of final state $f_{\text{final}}(x)$.

Following Newton and the interpretations of Newton's mechanics in the 19th century, the axioms of mechanics had been analytically represented in terms of an equation of motion whose solutions are time dependent functions. Selection problems are solved by the application of Maupertuis' principle of least action whose relevance consists in the choice of a certain path from the set of all possible paths. However, as it had been discussed in Chap. 7, Helmholtz derived the concept of work by the treatment of *paths* and *configurations* on an equal footing studying a mechanical system which returns after a certain time interval into the *initial configuration*. The intermediate configurations are not explicitly described. The equation of motion is *not susceptible* to a certain configuration, but is assumed to be valid for any configuration the system can attain.[21]

[20] The change of velocity is independent of the velocity (compare Chap. 4).

[21] Therefore, the initial values for position and velocity can be chosen arbitrarily. The equation of motion cannot depend on the total energy since this quantity depends on the special choice of the initial values.

In 1923, applying the same procedure, Schrödinger analyzed the motion on an *electron* which returns into its initial position [Schrödinger, Quantenbahnen]. Referring to Weyl's analysis of space, time and motion [Weyl, Raum und Zeit], Schrödinger obtained an exponential function $\exp\lfloor i(e/\hbar)\oint \vec{A}\cdot d\vec{s}\rfloor$ whose argument is made up of the electron charge e, the action parameter $\hbar$ and the vector potential $\vec{A}$. Hence, the distinction between configurations and paths becomes crucial.

In 1933, Schrödinger analyzed the relations between classical and quantum mechanics in terms of the different treatment of paths. Applying Maupertuis' principle in classical mechanics, a *selection* problem had been defined and solved since one and only one path is chosen from the variety of all possible paths. The other paths are excluded [Schrödinger, Nobel Lecture].[22] Such an interpretation had been also given for the magnitude of forces appearing in the interaction of bodies. Euler claimed that these forces are the least forces[23] which are necessary to avoid the penetration of bodies [Euler E344, Lettre LXXXVIII].[24]

However, assuming that the system is described by the energy, the paths cannot be related to all possible configurations of the system since the invariant value of the total energy truncates the paths of bodies whose continuation would violate the energy conservation. This behaviour can be readily demonstrated for constant total energy $E = T(p) + V(x) = \text{const}$ in case of motion along a straight line for a harmonic oscillator. It is obvious that the maximum of potential energy $E = V_{\max}(x)$ is attained for the maximal elongation $x_{\max}$ of the mass where simultaneously the kinetic energy is equal to zero. Nevertheless, the equation of motion $dv = (K/m)dt$ remains to be valid beyond this distance because of its independence of initial values and energy. Hence, the selection problem of paths has to be formulated differently in case of *given forces* and in case of *given energy*. Introducing a wave function for a certain energetic state of the system which is defined in the *whole* configuration state [Schrödinger, Heisenberg, p. 736], it is impossible to truncate the path without truncating the wave function, too. From Schrödinger's dictum, it follows: either the wave function does *not* describe a *path* if it is defined in the whole configuration space or the wave function describes a path, than it cannot be defined in the whole configuration space. Consequently, Schrödinger's analysis of the relations

[22] "We are faced here with the full force of the logical opposition between an either – or (point mechanics) and a both – and (wave mechanics). This would not matter much, if the old system were to be dropped entirely and to be *replaced* by the new. Unfortunately, this is not the case." [Schrödinger, Nobel Lecture]

[23] "(...) the impenetrability of both acts equally: and it is by their joined operation that the force necessary to prevent the penetration is supplied; we then say that they act upon each other, and that the forces resulting from their impenetrability produces this effect. (...) You will find here, therefore, beyond all expectation, the foudation of the system of the late *Mr. de Maupertuis*, so much cried up by some, and so violently attacked by others. His principle is, that in all changes which happen in nature, the cause which produces them is the least that can be." [Euler E344, Lettre LXXVIII]

[24] Therefore, the *paths* are automatically chosen since the shape of the path depends on the magnitude of forces.

between classical mechanics and quantum mechanics remains to be still in power if the "paths" are replaced with the "configurations" of a system.[25]

8.4 Schrödinger's Approach: Configurations and States

All essential mathematical tools for modern 20th century physics had been developed in 19th century. The mathematics of the general theory of relativity was created by Bolyai, Gauß and Riemann several decades before Einstein. The basic differential equation, later used for the description of quantum mechanical harmonic oscillator, was already introduced by Weber in 1869 and treated by Whittaker in 1903 [Whittaker/Watson, Chap. XVI]. The solutions of the equation in terms of polynomials had been presented by Hermite in 1864 [Whittaker/Watson, p. 350].

In contrast to Einstein's general theory of relativity, the development of quantum mechanics (QM) was not directly guided by mathematics. However, Schrödinger recognized the geometrical representation of motion by Hamilton and acknowledged the attempts of Felix Klein in 1891 to stimulate the physics community to make use of Hamilton's theory. The response to Klein's proposals was disappointing [Schrödinger, Second Announcement]. Some decades later, Schrödinger re-discovered Hamilton's theory as a link between mechanics and optics and, not surprisingly, succeeded in developing the basic equation of quantum mechanics independently of Heisenberg. Hence, in the 20th century, the communication between mathematicians and physicists did not flourish as in ancient times 200 years ago[26] when Euler made immediate use of his mathematics for classical mechanics and coordinated his progress in mathematics with his progress in physics.

In his program for mechanics [Euler E015/016] (compare Chap. 4), Euler distinguished between (a) bodies of infinitesimal magnitude and (b) bodies of finite magnitude. The bodies described in frame (a) are nowadays known as mass points. Euler introduced a general law for mechanics [Euler E177], [Euler E289] which is based on the assumption of translational motion of bodies of infinitesimal magnitude. The set (b) comprises all *non-infinitesimal bodies* which cannot be treated as mass point since their motion is a combination of translations and rotations.

The mechanical system is characterized by constitutive or material parameters which are always given in terms of finite numerical values. Space and time are not

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[25] Then, reconsidering Schrödinger's statement, it follows: "We are faced here with the full force of the logical opposition between an either set I of configurations – or set II of configurations (point mechanics) and a both set I of configurations – and set II of configurations (wave mechanics). This would not matter much, if the old system were to be dropped entirely and to be *replaced* by the new. Unfortunately, this is not the case." [Schrödinger, Nobel Lecture] It is assumed that set I and set II together form the whole configuration space.

[26] Regretfully, Weyl commented: "Ich kann es nun einmal nicht lassen, in diesem Drama von Mathematik und Physik – die sich im Dunkeln befruchten, aber von Angesicht zu Angesicht so gerne einander verkennen und verleugnen – die Rolle des (wie ich genügsam erfuhr, oft unerwünschten) Boten zu spielen." [Weyl, Group Theory] [http://www.math.uni-hamburg.de/math/ign/hh/biogr/weyl.htm]

considered as material parameters, therefore, they are included in the theory as finite or infinitesimal quantities. Planck introduced finite differences between energies of the states i and k which are described by the action parameter $\Delta E = E_i - E_k = \hbar \cdot \omega$.[27] The frequency ω depends on the material parameters mass and force constant.

8.4.1 Euler's Mechanics Reconsidered

In mechanics, the fundamental constitutive parameter is the inert mass m. The inert mass m of an infinitesimal body may be described by the same quantity as the inert mass of an extended body, since both models are considered as mechanical systems. The only difference is that total mass of the finite body is obtained by the integral $M = \int dm$ taken for the space region the body is occupying [Euler E289]. It is expected that the energy conservation law holds not only for any of the considered systems of different type, but it is also valid for the interaction of systems of different types.[28] In all three cases, it is necessary to introduce a purely coordinate dependent function. Furthermore, it will be demonstrated that Euler's procedure is also appropriate for the analysis of the relations between classical mechanics and quantum mechanics. In the second decade of 20th century, the development of quantum mechanics began with the rejection of basic concepts of classical mechanics. Heisenberg rejected the paths [Heisenberg 1925] and Schrödinger intended to replace point mechanics with an "undulatory mechanics". [Schrödinger, Second Announcement]

Euler's division of bodies into different types is based on the mathematical distinction between infinitesimal and finite quantities. The assumed set of mathematical quantities is not complete so far. It has to be supplemented by *infinite* mathematical quantities, since only *infinitesimal* and *infinite* quantities are complementing each other properly [Euler E212].

Following Euler, the basic distinction is made

- first between *infinitesimal* and *non-infinitesimal* magnitudes, denoted as set (A) and set (non-A), respectively, and,
- second between *finite* and *infinite* magnitudes, denoted as set (B) and sets (C), respectively.

Establishing and realizing his program for mechanics, Euler demonstrated that a finite parameter called *mass* can be assigned to an infinitesimal body [Euler E015/016].

Then, a significant question follows directly from Euler's program: Is it possible to assign a *finite* constitutive parameter to a system of *infinite* extension?

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[27] By the same procedure, Bohr calculated characteristic *energy* and *length* for the hydrogen atom using the electric charges of electron and nucleus, the mass of the electron and the action parameter [Bohr 1 to 4], $E = me^4/\hbar^2$ and $a = \hbar^2/me^2$, respectively.

[28] An instructive example is the photoelectric effect, which has been explained by the interaction of point like light quanta and bodies of finite size [Einstein, Heuristisch].

The answer is "yes" if we start with a purely mathematical definition of the density related to any of the constitutive parameters of the system. The definition of the densities related to bodies in set (B) is valid for different types of densities $\rho_b^{hom}(x)$ and $\rho_c^{non-hom}(x)$, being either homogeneous or non-homogeneous, respectively. However, there is a striking difference between these two basic types of densities as far as the extension of the system is concerned. Assuming a finite homogeneous density, the integral $M = \int_{-\infty}^{+\infty} dx \rho_{hom}(x)$ is necessarily divergent. Therefore, it may be expected that systems of type (C) are to be described by functions whose coordinate dependence may be related to non-homogenous densities.

The procedure is continued as follows. Euler's method for the definition of relations between (A) and (B) Systems will be transferred to the analysis of the relations between (B) and (C) Systems. At first, Schrödinger's procedure is reconsidered and reinterpreted in terms of Euler's methods.

Following the general procedure introduced by Newton and Euler, we reconsider Schrödinger's approach which is essentially based on the introduction of the wave function $\psi(x)$. This function is related to the internal energetic states of the system described by the series of energies (compare Sect. 8.2.2). Therefore, the functions are indicated by a subscript $\psi_E(x)$. In case of harmonic oscillator or other quantum mechanical system described by a series of internal states $E_0, E_1, E_2, \ldots$, the energies E_i are not variables like the coordinates x or the time t, but *parameters* of the system. All these parameters are correlated with the *same* set of independent variables $-\infty < x < +\infty$ since the system is not pushed into new configurations by the transition into *another* energetic state.[29] Following Schrödinger, the investigation of the states $E_0, E_1, E_2, \ldots$ can be performed for the non-interacting system. Hence, it is assumed that the system does not interact with the environment and the internal energy is not changed. The system is not translated in space and it does not rotate about an axis.

8.4.2 *Energy and Configurations*

Schrödinger considered solutions $\psi_E(x)$ of the time-independent wave equation [Schrödinger, Second Announcement]. The function $\psi_E(x)$ has been assumed to be defined and to exist for all configurations of the system. Then, it is expected that *different* states are described by *different* wave functions since the configurations are modified neither by an increase not by a decrease of the internal energy of the system. This is contrary to the classical case where the analytical expression for the internal energy $E = T(p) + V(x)$ is independent of the magnitude of the energy. The change is due to different configurations expressed by different sets of position and momentum being the maximal elongation and the maximal velocity in case of the linear harmonic oscillator, i.e. $-x_{\max} < x < +x_{\max}$ and $-p_{\max} < x < +p_{\max}$, respectively.

[29] This behaviour is essentially different from classical systems (compare Chap. 7).

As in Eulerian mechanics, where the change in velocity dv had been related to the mass m and the forces K (compare Chap. 4), here, total energy E and the coordinate dependent potential energy $V(x)$ are related to the function $\psi_E(x)$. Both functions are required since the total energy is independent of coordinates and the potential energy is independent of total energy. If this function is related to the energy

$$\frac{1}{E - V(x)} \sim \psi_E(x), \tag{8.4}$$

Schrödinger's criterion of finiteness is satisfied for large values of x. Obviously, the function $\psi_E(x)$ cannot be calculated using Eq. (8.4). This relation is as incomplete as the relation $dv \sim K/m$ was (compare Chaps. 2 and 4). Therefore, we have to complete the description of the system adding kinetic energy $E = T(p) + V(x)$ where $T(p)$ is the kinetic energy as a function of momentum. Therefore, assuming Schrödinger's wave function as the basic quantity, we have also to express the kinetic energy in terms of the function $\psi_E(x)$. Nevertheless, there is an essential difference in comparison to Schrödinger's approach since the quantity E in Eq. (8.4) is not necessarily confined to the *discrete* set which is observed experimentally and, consequently, the function $\psi_E(x)$ is not necessarily related to one of the states of the system. Following Einstein [Einstein, Wärme] in establishing the reduced set of energies, the physically formulated selection problem is to be completed by a mathematical version where, as in case of physics, both the sets of energies, the continuous and the discrete one, have to be investigated before any choice can be made. Hence, beside physical solutions, Schrödinger's equation should also provide non-physical mathematical solutions. Before this will be done in the following section, the results of the discussion are summarized.

Schrödinger's approach completes and extends in a quite natural way Euler's program for mechanics. The systems are distinguished according to their occupation of space regions of different magnitude in configuration space. Then, the following types of theory in dependence on the type of extension can appear:

(I) the theory of bodies of infinitesimal magnitude, Euler,
(II) the theory of bodies of finite magnitude, Euler,
(III) the theory of bodies or systems of bodies of infinite magnitude, Schrödinger.

Item (III) should be related to Schrödinger's assumption that the function $\psi(x)$ is defined in the whole configuration space. Obviously, the topics (I) to (III) comprise all possible cases of those mathematical quantities which Euler had defined within the mathematical frame [Euler E387/388], [Euler E212]. Euler defined mathematical quantities as being susceptible to diminishing and increase. Following Euler, the existence of such function is primarily a mathematical question. Euler's procedure is to relate infinitesimal quantities to finite quantities, i.e. numbers of different type to each other. Now, this procedure is transferred to the problem to relate infinite quantities to finite quantities.

After solving the mathematical problem, the physical problem has to be answered whether the numbers are related to objects by measurement. Obviously, all physical systems are finite. Therefore, it is impossible to confirm the theory by the measurement of all quantities, since infinitesimal and infinite quantities cannot be measured.[30]

The following procedure results directly from Euler's theory of a body of infinitesimal magnitude. Euler introduced this type of bodies by the modification and reinterpretion of the *geometric* characteristics (extension, shape) preserving the *physical* parameters, the mass and the path. The geometric connection between two given points is replaced with the mechanical connection of two positions by the path of the body and vice versa. The significant difference between a mechanical model and a purely geometrical model of the body is due to inertia, time and forces. Any distance is subdivided by a uniformly moving body into equal parts and, simultaneously, the time interval assigned to the whole path is also subdivided into equal parts (compare Chap. 4). The connection between two different places in space is geometrically given by their distance. This distance can be determined experimentally without the motion of a body whose path contains the two given points. Therefore, the configurations are always defined prior to the path. A configuration is defined for a system of resting bodies which do not change their positions. The positions are given by a set of coordinates $x_i, i = 1,2,3,\ldots,n$, the path is given by an ordered set $x_1, x_2, x_3, \ldots, x_n$ of coordinates using an external ordering parameter $t_1 < t_2 < t_3 < \ldots < t_n$ called time. Then, the path can be represented as a function $x = x(t)$ which is parametrized by time. The same parametrization is assumed for the momentum $p(t)$. The exclusion of all other configurations except those which are belonging to the path is described by the delta functions

$$\rho(x) = \delta(x - x(t)), \quad \sigma(p) = \delta(p - p(t)) \tag{8.5}$$

for the path, i.e. the trajectory in configuration space, and the trajectory in momentum space, respectively, which are only different from zero if the arguments vanish, i.e. for $x = x(t)$ and $p = p(t)$. This procedure relates the parameterization by time to a certain position, i.e. a configuration, the body is occupying. The position is represented by a geometrical point which is in agreement with Euler's assumption on the theory of infinitely small bodies. Then, any body travelling along this path must be considered to be infinitesimal if it touches only those configurations (positions) which are defined by a geometric line. These relations will be discussed more in detail for the commonly used model system, the linear harmonic oscillator. The *continuity* of the path is included into the argument of the *discrete* delta-functions by the functions $x = x(t)$ and $p = p(t)$. Although the algorithms for delta-function had been completely founded only in the 20th century, Euler was right in assuming such objects like mass points with an infinite mass density.

[30] Usually, it is said that some of the relations are only approximately valid and the neglecting of a small quantity in comparison to a large quantity is equivalent to the relation between an infinitesimal and a finite physical quantity, as Leibniz claimed (compare Chap. 3). However, this model does not work as far as the relations between infinitesimal quantities are concerned since, following Euler, the geometric ratio of two infinitesimal quantities is finite [Euler E212, § 85]. This ratio cannot be obtained experimentally by measuring both the quantities separately.

The relation to energy is readily obtained and will be demonstrated for a model system, the linear harmonic oscillator, whose energy is given by $E = T(p) + V(x)$ for $0 \leq E < +\infty$ in classical mechanics. The expression is valid for any configuration and any momentum, i.e. $-\infty < x < +\infty$ and $-\infty < x < +\infty$, respectively.

The results of the previous section remain to be valid if we introduce a purely coordinate dependent function $V(x)$ instead of the special expression for the potential energy of the harmonic oscillator. The expression for the kinetic energy $T(p)$ remains to be the same as before since it is independent of the special system. Using the delta functions given by Eq. (8.5), the energy is related to the paths by the formula

$$E_{\text{path}} = \frac{\int_{-\infty}^{+\infty} dp\delta(p - p(t))T(p)}{\int_{-\infty}^{+\infty} dp\delta(p - p(t))} + \frac{\int_{-\infty}^{+\infty} dx\delta(x - x(t))V(x)}{\int_{-\infty}^{+\infty} dx\delta(x - x(t))} \tag{8.6}$$

since the integral $\int dx\delta(x - x(t)$ is only different from zero for those space regions the body is moving along the path. Hence, there is an automatic truncation for those times which are not parametrizing the positions of the path. The same truncation takes place for the momentum integrals. The result is

$$E = E_{\text{path}} = T(p(t)) + V(x(t)) = \text{const} \tag{8.7}$$

and the energy is now related to the path of the moving body. Then, the energy takes a definite value which can be only modified by the interaction of the system with other bodies or systems. The total energy becomes a parameter of the system. All the states of the system are characterized by functions of the *same* shape.

The corresponding procedure is now applied to a non-interacting system having also *different* energetic states, but those which are described by *different* coordinate dependent functions (compare Table 8.5). As before, the states can be only changed due to an interaction. Nevertheless, the states cannot be specified by only one equation like Eq. (8.7) because of their multiplicity. Hence, a certain state of the system should be represented in terms of Schrödinger's wave function $f_{state}(x)$ and the corresponding momentum dependent function $g_{state}(p)$. The integration is performed in the whole configuration space and the corresponding momentum space.

$$E_{\text{state}} = \frac{\int_{-\infty}^{+\infty} dp\,|g_{\text{state}}(p)|^2\,T(p)}{\int_{-\infty}^{+\infty} dp\,|g_{\text{state}}(p)|^2} + \frac{\int_{-\infty}^{+\infty} dx\,|f_{\text{state}}(x)|^2\,V(x)}{\int_{-\infty}^{+\infty} dx\,|f_{\text{state}}(x)|^2} \tag{8.8}$$

In Eq. (8.8) the energy E_{state} and the functions $f_{\text{state}}(x)$ and $g_{\text{state}}(p)$ are related to the *same energetic state* of the system. Hence, the states of any other system should be described by another energy and, consequently, by functions of another shape. The expression $|g_{\text{state}}(p)|^2$ in the integrand is chosen as a positive quantity because of the previously assumed conditions $0 \leq E < +\infty$, $T(p) \geq 0$ and $V(x) \geq 0$ for the model system. Already at this stage, it can be concluded that the essential difference between the classical and the non-classical system can be represented by the condition that, for the latter system, there is no such an energetic state being characterized by $E = 0$ where the total energy is equal to zero. The reason is that both terms in Eq. (8.8) are greater than zero and disappear only in the special case where $T = 0$ and $V = 0$ are the only parameters of the system. Hence, instead of the previous condition $0 \leq E_{\text{class}} < +\infty$ we have now

$$E_{\text{non-class}} = E_{\text{state}} > 0. \tag{8.9}$$

However, the state $E = 0$ is not indeterminate, but it can be singled out by applying the method of maxima and minima. Following Leibniz, Euler and Maupertuis, the minimum of $E = T(p) + V(x)$ is given by the sum of the minimal values of $V(x)$ and $T(p)$ which are determinate by the conditions

$$\frac{\partial V}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial T}{\partial p} = 0. \tag{8.10}$$

For the harmonic oscillator, potential and kinetic energy are given by the functions $V(x) = (k/2)x^2$ and $T(p) = (1/2m)p^2$, respectively. Hence, using Eq. (8.10) it follows $E_{\text{min}}^{\text{class}} = T(0) + V(0) = 0$ and, straightforwardly, $E_{\text{min}}^{\text{non-class}} > 0$. Now, the minimal energy is set equal to the parameter E_0 equation in Table 8.3, i.e. $E_0 = E_{\text{min}}^{\text{non-class}}$. As it had been discussed above the minimal energy cannot be experimentally determinate. $E_0 = E_{\text{min}}^{\text{non-class}}$ is an unknown parameter whose dependence on the parameters of the system is now specified.[31] Hence, its value should be obtained from Eq. (8.8) if the representation is *necessary* and *sufficient* for the calculation of the energetic spectrum of the harmonic oscillator.

8.4.3 Quantization as Selection Problem

Schrödinger intended to replace "point mechanics" by an "undulatory mechanics" and interpreted the former in Newton's frame established by an equation of motion of d'Alembert type divgrad $\psi - u^{-2}\partial^2\psi/\partial t^2 = 0$ for coordinate and time dependent functions $\psi = \psi(x,\ t)$ [Schrödinger, Second Announcement]. Therefore, the interpretation of the results had been given in terms of models for time-dependent

[31] In the first version of the theory of black body radiation, Planck assumed $Emin = 0$ for the harmonic oscillator which had been later replaced with $E_{\text{min}} = \hbar\omega/2$ by thermodynamical reasons [Planck, Vorlesungen, §§ 137 and 138, 184]. Planck derived the formula $E_n = (n - 1/2)h\nu$, hence for $n = 1$ it follows $E_1 = h\nu/2$. However, Planck interpreted the result in terms of arithmetical mean value between the energies $(n-1)h\nu$ and $nh\nu$, or, the "mittlere Energie eines im n. Elementargebiet befindlichen Oscillators" [Planck, Vorlesungen, § 138]. Analytically and independently of thermodynamics, the latter version $E_{\text{min}} = \hbar\omega/2$ was only confirmed by Heisenberg [Heisenberg 1925] and Schrödinger [Schrödinger, First to Fourth Announcement] as the consequence of the new approach. In 1925, Heisenberg compared the "old quantization" and his approach to each other and commented: classical quantization (old quantum theory), $W = n\hbar\omega$, quantum theoretical quantization $W = (n + 1/2)\hbar\omega$, for $n = 0, 1, 2, 3, \ldots$ [Heisenberg 1925, p. 889]. "$(\ldots)$ denn die $(\ldots)$ zunächst unbestimmte Konstante wird durch die Bedingung festgelegt, daß es einen Normalzustand geben solle, von dem aus keine Strahlung mehr stattfindet ...)." [Heisenberg 1925, p. 886]

wave propagation. The momentum dependent functions $g_{\text{state}}(p)$ have not been considered from the very beginning by Schrödinger.

Now, remembering Schrödinger's dictum "(NB. nicht '*pq*-Raum', sondern '*q*-Raum'.)" [Schrödinger, Heisenberg, p. 736][32] and completing the statement by "(NB. nicht '*pq*-Raum', sondern '*p*-Raum'.)", the differential equation called by Schrödinger "amplitude equation" [Schrödinger, Fourth Announcement, p. 110] has to be represented either by coordinate or by momentum dependent functions. Hence, either momentum or coordinate dependent function had to be eliminated from Eq. (8.8). For that purpose, a relation between the functions $f_E(x)$ and $g_E(p)$ is established which is *independent* of the state or the energy, but relates functions belonging to the *same* state to each other *independently* of the magnitude of parameter E.

$$f_E(x) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} e^{-i\frac{xp}{\alpha}} g_E(p)dp \tag{8.11}$$

$$g_E(p) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} e^{i\frac{px}{\alpha}} f_E(x)dx. \tag{8.12}$$

The relations (8.11) and (8.12) are valid, if all configurations of the system are taken into account and the energy conservation law is not violated. Additionally, a new parameter, α, having the dimension of action had to be introduced for dimensional reasons. By definition, the parameter is *independent* of x and p and, it is assumed to be independent of energy. The argument of the exponential function is a pure number. In the general case, the functions $f_E(x)$ and $g_E(p)$ may also become complex valued and it holds $|f_E(x)|^2 = f_E^*(x)f_E(x)$ and $|g_E(p)|^2 = g_E^*(p)g_E(p)$. Then, eliminating the momentum dependent functions the following relation is obtained

$$\int_{-\infty}^{+\infty} dx f_E^*(x) \left[E f_E(x) - V(x) f_E(x) + \frac{\alpha^2}{2m} \frac{\partial^2 f_E(x)}{\partial x^2} \right] = 0. \tag{8.13}$$

The expression in the brackets $[F(x,E)]$ can be chosen differently, either being different from zero $[F(x,E)] \neq 0$ or equal to zero $[F(x,E)] = 0$. In the latter case, a relation between the energy E and the function $f_E(x)$ is established which is valid for *each* of the configurations of the system.

$$\frac{\alpha^2}{2m} \frac{\partial^2 f_E(x)}{\partial x^2} + (E - V(x)) f_E(x) = 0. \tag{8.14}$$

This relation replaces the Newton-Euler equation of motion which is also valid for each of the configurations of the system. However, the difference is that the energy is now included as an indispensable parameter of the theory independently of the

[32] The advantages of Schrödinger's coordinate dependent functions in comparison to matrix mechanics had been readily demonstrated in calculating the spectrum of the hydrogen atom. Pauli made use of matrix mechanics and had to perform cumbersome calculations [Pauli].

considered configuration. Identifying α with the action parameter $\hbar$ and assuming the model of the harmonic oscillator we obtain the stationary Schrödinger equation (or "amplitude equation"), where $\omega = \sqrt{k/m}$ is defined as in classical mechanics and is identical with the frequency of Planck's oscillator. The expression in the brackets can be chosen differently, either being different from zero or equal to zero. In the latter case, a relation between the energy and the function is established which is valid for each of the configurations of the system.

Now, specifying the potential energy $V(x)$, the functions and the energies are determinate by this equation. Eq. (8.13) can be differently interpreted, (i) the whole expression is necessarily taken into account to fulfil the requirement to vanish or (ii) only the integrand is chosen such that it vanishes. In the latter case, two differential equations

$$\frac{\alpha^2}{2m}\frac{\partial^2 f_E(x)}{\partial x^2} + (E - V(x))f_E(x) = 0 \tag{8.15}$$

$$\frac{\alpha^2}{2m}\frac{\partial^2 f_E^*(x)}{\partial x^2} + (E - V(x))f_E^*(x) = 0 \tag{8.16}$$

for the energies and the functions $f_E(x)$ and $f_E^*(x)$ are obtained. The procedure is completed by eliminating the *coordinate* dependent function in Eq. (8.8) instead of *momentum* dependent functions. Then, we obtain a differential equation in momentum space for the same energies without violating principles postulated by Schrödinger. The only modification is that Schrödinger's statement about exclusion of the phase space: "(N.B. nicht 'pq-Raum', sondern 'q-Raum')" has to be replaced with the statement "(NB. nicht 'pq-Raum', sondern 'p-Raum)" which was already implicitly embodied in Schrödinger's assumption. Then, quantization can be defined as a selection problem [Suisky 2001], [Enders 2004], [Enders 2005], [Suisky 2007a]. The difference between the representations does only emerge in the time dependent wave equation since the configuration space function $V(x)$ is replaced with the time dependent function $V(x,t)$ whereas $T(p)$ is not altered by the introduction of time dependence [Schrödinger Fourth Announcement].

The selection problem is properly defined by the operation to chose a number or a set of numbers from the set of real numbers.[33] However, integers as mathematical numbers are only obtained after the introduction of dimensionless quantities for all quantities appearing in Eq. (8.15) and (8.16). Specifying the potential energy as the only function which characterizes the physical system until now, $V(x)$ will be chosen for a model system, the linear harmonic oscillator, as $V(x) = (k/2)x^2$ and the frequency $\omega_0 = \sqrt{k/m}$. Then, the dimensionless variables are obtained

[33] In terms of energy, this type of problem was introduced by Einstein in 1906 who claimed that the number of energy states of a *molecular* body is *less* than the number of states of bodies of our *sensual experience* [Einstein, Wärme]. From Schrödinger's approach it follows that the problem has to be formulated in terms of real numbers. The integers should be obtained quite naturally as a special subset without imposing a "condition for quantization in terms of integers" [Schrödinger, First Announcement].

$$\xi = \sqrt[4]{\frac{4 \cdot k \cdot m}{\alpha^2}} x; \; \eta = \sqrt[4]{\frac{4}{k \cdot m \cdot \alpha^2}} p; \; \nu \equiv \frac{E}{\alpha \omega_o} - \frac{1}{2} \tag{8.17}$$

and the differential equation reads as follows[34]

$$\frac{d^2 D_\nu(\xi)}{d\xi^2} + \left(\nu + \tfrac{1}{2} - \tfrac{1}{4}\xi^2\right) D_\nu(\xi) = 0. \tag{8.18}$$

The mechanical background is implicitly preserved, but the problem had been represented in term of mathematical relations between numbers. All the restrictions previously introduced by mechanical reasons are now removed and the parameter ν is decoupled from the condition for energy $0 \leq E < +\infty$ and can be varied in the whole interval of positive and negative real numbers $-\infty < \nu < +\infty$. Hence, it can be expected that the selection problem should be formulated in terms of relations between different sets of real numbers. The different sets are to be related to the corresponding sets of functions of different type. The classification of functions can only be successfully performed if the *complete* set of mathematical solution is known. As usual, the complete set contains *physical* and *non-physical* solutions. Schrödinger emphasized that the solutions of preferential interest are those which are finite in the whole configuration space.[35] Nevertheless, mathematicians were interested in the construction of the complete set of solutions of the differential equation (8.18) which is known in the mathematical literature as Weber's equation of the parabolic cylinder [Whittaker/Watson, Chap. XVI]. The solutions were studied by Whittaker in 1903. The variable ν is defined in the whole interval $-\infty < \nu < +\infty$. However, any selection of special parameter values can be performed by a procedure which introduces relations between different values of the parameter. Applying Whittaker's method, the general solution can be represented as a coupled set of first order differential equations, usually known as recurrence relations

$$\frac{dD_\nu(\xi)}{d\xi} + \frac{\xi}{2} D_\nu + \nu D_{\nu-1}(\xi) = 0. \tag{8.19}$$

$$\frac{dD_\nu(\xi)}{d\xi} - \frac{\xi}{2} D_\nu + \nu(\nu+1) D_{\nu+1}(\xi) = 0. \tag{8.20}$$

which substitutes for Eq. (8.18). The remarkable property of this set of coupled equations is that using Eqs. (8.19) and (8.20) the whole set of parameter values can be obtained by the choice of any arbitrary parameter value taken from the interval

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[34] For the definition of the functions $D_\nu(\xi)$ compare Whittaker [Whittaker/Watson, Chap. 16] and Abramowitz M and Stegun I A (eds) *Handbook of mathematical functions* (1964) National Bureau of Standards, Applied Mathematics Series 55

[35] "Denn es zeigt sich in allen Fällen der klassischen Dynamik, (...) daß die Gleichung (...) *selbsttätig* gewisse Frequenzen und Energieniveaus als die für die stationären Vorgänge möglichen aus[sondert], ohne irgendeine weitere Zusatzannahme als die für eine physikalische Größe beinahe selbstverständliche Anforderung an die Funktion ψ: dieselbe soll im ganzen Konfigurationsraum eindeutig endlich und stetig sein." [Schrödinger, Second Announcement, p. 511]

$-1 \leq v \leq 0$. The indices of any two functions being correlated with the function of a chosen v are always determinate by the differences $\Delta v = \pm 1$.

The selection of a certain set of numerical values can be only performed by selection criteria which are related to the recurrence relations. These relations comprise the full information about the system including the mutual relation between different states. In contrast to the usual procedure, the coordinate dependent functions are not explicitly calculated for each of the state, but the whole set of parameter values is divided into different subsets which belong to the corresponding subset of functions.

The following selection criteria are introduced:
Selection criterion. The properties of functions. Maxima and minima: The difference between border points and internals points follows from the decoupling of the system of equations for $v = 0$ and $v = -1$.

$$\frac{dD_0(\xi)}{d\xi} + \frac{\xi}{2} D_0 = 0 \tag{8.21}$$

$$\frac{dD_{-1}(\xi)}{d\xi} - \frac{\xi}{2} D_{-1} = 0. \tag{8.22}$$

Therefore, three types of solutions are obtained belonging either to the *border* points or to the *internal* points of the interval and the whole interval is subdivided into solutions belonging to (i) $-1 < v < 0$, (ii) $v = -1$ (Eq. 8.22) and (iii) $v = 0$ (Eq. 8.21). The subsets of functions can be properly distinguished mathematically by minima and maxima or by divergent solutions in case of Eq. (8.22) and non-divergent solutions in case of Eq. (8.21).

$$D_0 = \exp\left(-\xi^2/4\right) \qquad \text{and} \qquad D_{-1} = \exp\left(+\xi^2/4\right). \tag{8.23}$$

Thus, a physical selection criterion can be applied to exclude divergent solutions as *non-physical* solutions. Additionally, the physical interpretation of the parameters in terms of energies is taken into account.

In terms of energy, the $v = 0$ set comprises positive values $E_0 = (1/2)\alpha\omega$, $E_1 = (3/2)\alpha\omega$, $E_2 = (5/2)\alpha\omega, \ldots$ (system I) whereas the $v = -1$ is made up of only negative values $E_{-1} = -(1/2)\alpha\omega$, $E_{-2} = -(3/2)\alpha\omega$, $E_{-3} = -(5/2)\alpha\omega$, $\ldots$(system II). In both cases, the interaction between the system and the environment is not only governed by the same rule, but it is also impossible to distinguish what kind of system is interacting with the environment. Phenomenologically, in any elementary step of absorption or emission, the environment supplies the system with the same quantum of energy $\Delta E = \pm\alpha \cdot \omega$ which it receives from the system in case of emission. Emission and absorption are not different in magnitude, but are only distinguished by the sign of ΔE which is assumed to be positive in case of absorption and negative in case of emission. This distinction corresponds to the supply and lost of energy according to the same principles which had been discussed by Helmholtz to find the proper sign of potential energy (compare Chap. 7). Hence, a distinction between the systems described by Eqs. (8.21) and (8.22) is impossible if only the elementary acts of emission and absorption are analyzed. However, a selection criterion is provided by the exclusion of perpetual motion.[36]

Selection criterion. Perpetuum mobile excluded: There are two possible interactions between system and environment: first, an unlimited supply of energy from the environment to the system; and second, an unlimited supply of energy from the system to the environment. The second case is has to be excluded by stating the *impossibility of perpetual motion*. In the first case, the supply of energy from the system to the environment is automatically limited due to the existence of the state having the lowest energy. Obviously, the supply of energy to the environment is not limited for a system whose lowest state is not limited since the series of energies can be decreased to minus infinity.[37]

Now, the whole procedure is finished. A *countable infinite* set of states, represented by the set of integers, had been defined for the system and chosen from the set of real numbers. This set cannot be further subdivided by physical or mathematical reasons into subsets. Therefore, neither additional mathematical nor additional physical problems are to be solved nor *additional parameters* are to be introduced. Hence, it can be concluded that there are no hidden parameters. Knowing the wave function D_0 and the energy E_0 of the ground state, the wave functions and the corresponding energies of all other states are obtained from Eq. (8.19). The whole procedure is *necessary* and *sufficient* for selecting the only countable infinite set of energy values having a *smallest* element and, simultaneously, the set of corresponding wave functions. In addition, this solution is the only physical solution. Therefore, the selection problem had not only been formulated, but also completely solved.

The basic idea to formulate quantization as selection problem had been traced back to Einstein's postulate of the specific features of molecular aggregates introduced in 1906. In 1926, although Schrödinger invented a new approach and treated quantization as an *eigenvalue problem*, essential characteristics of a selection procedure remained to be implicitly included. Schrödinger claimed that the advantage of the postulated differential equation is to single out ("sondert aus") special energetic levels for the investigated system as the only ones which are physically possible: "Die Gleichung (...) sondert (...) selbsttätig gewisse Frequenzen oder Energieniveaus als die (...) allein möglichen aus, (...)." [Schrödinger, Second Announcement] Thus, this *special* set of solutions is to be related to the *whole* set of solutions of the differential equation, i.e. to those solutions which are mathematically possible. The subsequent subdivision of the latter ones into subsets is performed by (i) the application of *mathematically* defined criteria followed and completed by (ii) the application of *physical exclusion* principles like the exclusion of perpetual motion. Thus, the finally obtained solution is mathematically consistently formulated and physically justified and confirmed.

By this procedure, the derivation and reliability of quantum mechanical basic equation complies with Euler's consistently formulated mechanics which refers to

[36] A similar criterion had been already applied by Leibniz for the analysis of the conservation of living forces [Leibniz, Specimen, I (11)] (compare Chap. 2).

[37] Sometimes, it is said that such systems show perpetual motions of third kind.

perfectly adapted mathematical methods. Euler subdivided mechanics into a theory of bodies of infinitesimal magnitude and a theory of bodies of finite magnitude. Both theories are not only complementary, but interfere with each other and aid one another due the common underlying mathematical and physical principles. In general form, these principles had been mathematically and physically consistently formulated by Euler who made use of the full power of the legacy of Descartes, Newton and Leibnitz.

The Eulerian procedure is found to be implicitly renewed and generalized in Schrödinger's wave mechanics. Introducing the wave function which is related to systems extended in the whole configuration space, Schrödinger did a pioneering step as important as Euler did 200 years ago, who assumed mechanical quantities to be related to bodies of infinitesimal magnitude.

Summary

The tercentenary celebrations of Leonhard Euler's birthday in 2007 were most welcome to reconstruct not only the development of mechanics during the post-Newtonian period in the first half of the 18th century, but also to accentuate the extraordinary role Euler played in the reconsideration and solution of long-established and controversially discussed problems and the invention of new principles and methods.

The editor of the *Mechancica* summarized the progress Euler made by stating that Euler invented a completely new approach of rearrangement and redesigning of the theoretical representation of mechanics being also different from Newton's axiomatics in the *Principia*. Euler intended not only to replace the geometrical methods by analytical representations, but surpassed all his famous and distinguished predecessors and contemporaries. "Die Versuche, die *Newton* und *Johann Bernoulli*, *Varignon* und *Hermann* nach dieser Richtung hin gemacht hatten, stehen weit zurück hinter der Neugestaltung, die wir Euler verdanken, und die *Mechanica* bleibt, wie es Lagrange in seiner *Mécanique analytique* (1788) ausgesprochen hat, 'le premier grand ouvrage où l'Analyse ait été appliquée à la science du mouvement'." [Euler E015/016 (Stäckel)]

Euler's work on mathematics and physics was not only most influential in the 18th century, but also for the following development of both disciplines. The contemporary physics is formulated in terms of differential equations of those types and their generalizations which had been developed in complete and consistent analytical representation by Euler for the first time in 1736.

In this book, they decisive steps Euler did had been summarized and analyzed and the connections to the 20th century state of art in physics had been demonstrated. The interplay between physics and mathematics which appeared in the 18th century had been especially compared to the development of quantum mechanics between 1900 and 1930 and relativistic physics initiated by Einstein and Minkowski between 1905 and 1910.

(i) Euler successfully paved the way for the replacement of *geometrical* with *analytical* methods being simultaneously performed in *mechanics* and *mathematics* and demonstrated

(ii) that a successful and satisfactory solution of the problem is constrained by the precondition to attain the rigour in demonstrations known from ancient sciences, i.e. from static and geometry, respectively.

(iii) Euler did not restrain this substitution of methods to the physical part of Newton's mechanics, but included a reconsideration of the foundation of calculus whereby the theory underwent a considerable modification. Euler invented

(iv) the concept of infinitely small body in the treatise *Mechanica sive motus scientia analytice exposita* [E015/016] and, moreover,

(v) made simultaneously use of the Cartesian and Newtonian measure of the quantity of motion and the Leibnizian measure of living forces and

(vi) methodologically, of the Leibnizian principles of contradiction, sufficient reason and continuity and

(vii) postulated a hierarchy of mathematical quantities of infinitesimal, finite and infinite magnitude which is centered upon finite quantities, but limited neither in the order of consecutive levels of the infinitely small nor of the infinitely large [E212].

(viii) The rigorous analytical formulation of mechanical laws results not only in "true, but necessarily true relations" [E015/016] since Euler got rid of any approximations by the only, but decisive reference to the distinguished levels of the hierarchy of infinitesimal and finite quantities [E212]. Reconsidering Euler's original version, it can be demonstrated

(ix) that Euler's algorithms are strictly confirmed to be valid by nonstandard analysis.

In the end of the 19th century, Felix Klein acknowledged Euler's innovations in mathematics and described their essence as decisive steps toward the *arithmetization of mathematics*. Euler pioneered this development by the foundation of the *differential calculus* by the *calculus of finite differences*. Following Newton and Leibniz who invented the basic rules for arithmetical operations with differentials and finite quantities, Euler also pioneered the replacement of geometrical with arithmetical methods in *Mechanics* [E015/016] based on the earlier foundation of the *differential calculus* [Euler 1727], [Euler E212].

The Eulerian procedure is found to be renewed and generalized in Schrödinger's wave mechanics. By introducing the wave function which is related to systems extended in the whole configuration space, Schrödinger made a pioneering step as important as Euler did 200 year ago, who assumed mechanical quantities were related to bodies of infinitesimal magnitude.

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